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Possible numbers of ones in 0–1 matrices with a given rank

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We determine the possible numbers of ones in a 0–1 matrix with given rank in the generic case and in the symmetric case. There are some unexpected phenomena. The rank 2 symmetric case is subtle.

Keywords: 0–1 Matrix; Rank; Number of ones; Symmetric matrix

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1. Introduction

A 0–1 matrix is a matrix whose entries are either 0 or 1. Such matrices arise frequently in combinatorics and graph theory. It is known [1, p. 243] that the largest number of ones in an $n \times n$ nonsingular 0–1 matrix is $n^2 - n + 1$. Interpreting nonsingularity as full rank, we may ask further the question: What are the possible numbers of ones in a 0–1 matrix with given rank? We will answer this question in the generic case and in the symmetric case. The rank 2 symmetric case is subtle. Valiant [2] defined the rigidity $R_A(k)$ of a matrix A to be the minimal number of entries in the matrix that have to be changed in order to reduce the rank of A to less than or equal to k. So our work is along lower bounds on rigidity of explicit matrices. See [3]. In section 2 we prove the main results. In section 3 we give some examples.

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2. Main results

THEOREM 1 *Let k, n be positive integers with $k \leq n$. There exists an $n \times n$ 0–1 matrix of rank k with exactly d ones if and only if*

(i) *$d = xy$ for some integers x and y with $1 \leq x \leq n,\ 1 \leq y \leq n$ when $k = 1$;*
(ii) *$k \leq d \leq n^2 - k + 1$ when $k \geq 2$.*

Proof First note that any $n \times n$ 0–1 matrix of rank k has at least k ones and has at most $n^2 - k + 1$ ones. Thus the condition $k \leq d \leq n^2 - k + 1$ is always necessary. Throughout, we denote by $f(A)$ the number of ones in a 0–1 matrix A.

(i) Let A be an $n \times n$ 0–1 matrix of rank 1. Let α be any nonzero row of A. Then each row β is either equal to α or equal to 0. Suppose α contains y ones and A has x nonzero rows. Then $f(A) = xy$. The 'if' part is obvious.
(ii) It suffices to prove the 'if' part. The case $k \geq 3$ is covered by Theorem 4 (iii), which follows. So we need to prove only the case $k = 2$ here. Let $2 \leq d \leq n^2 - 1$. For every such d, we will exhibit an $n \times n$ 0–1 matrix of rank 2 with d ones. Let

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $f(B) = 2$. Next starting with B we increase the number of ones by one in each step, up to $n^2 - 1$. At the same time all these matrices are of rank 2, which can be seen by looking at the rows. Keeping the entry $B(2, 1) = 0$ fixed and successively changing the 0's in the first two rows to 1's, we obtain the matrix

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then in B_1 successively setting $B_1(i, 1) = 1,\ i = 3, 4, \ldots, n$ we obtain the matrix

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In general we denote by B_t the $n \times n$ 0–1 matrix whose first $t + 1$ rows consist of ones and other rows consist of zeros except that the first column has only one nonzero entry at $(1, 1)$ position, and denote by C_t the matrix obtained from B_t by making the first column all ones except that the $(2, 1)$ entry is 0, $t = 1, \ldots, n-1$. Thus

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that $f(B_2) = f(C_1) + 1$. Set $B_2(i, 1) = 1$ and retain the other entries for $i = 3, 4, \ldots, n$ successively. We obtain

$$C_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Observe that

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $f(B_3) = f(C_2) + 1$. Repeating the above process by using the last $n-2$ entries of the first column we finally obtain the matrix C_{n-1} whose only zero entry is $C_{n-1}(2, 1)$ and $f(C_{n-1}) = n^2 - 1$. This completes the proof. ■

Now we turn to the study of symmetric 0–1 matrices. To establish the second main result, we first prove two lemmas. Denote by $A[1, 2, \ldots, k]$ the principal submatrix of A lying in the first k rows and first k columns.

LEMMA 2 *Let A be a symmetric complex matrix with rank$(A) = k$. If the first k rows of A are linearly independent, then $A[1, 2, \ldots, k]$ is nonsingular.*

Proof In fact it is easy to show the following more general result: Let A be an $m \times n$ complex matrix with rank$(A) = k$. If the first k rows of A are linearly independent and the first k columns of A are linearly independent, then $A[1, 2, \ldots, k]$ is nonsingular. This follows since the rank remains unchanged after deleting the last $m - k$ rows of A, and then deleting the last $n - k$ columns of the resulting matrix. ■

LEMMA 3 *Let A be a symmetric* 0–1 *matrix with rank$(A) = 2$. Let α and β be two linearly independent rows of A and γ be any row of A. Then $\gamma = \alpha$, or $\gamma = \beta$, or $\gamma = 0$.*

Proof $\gamma = u\alpha + v\beta$ for some real numbers u, v. Since α and β are linearly independent, it is not hard to show that $u, v \in \{0, 1, -1\}$. By simultaneous row and column permutations if necessary, without loss of generality we may suppose that $\alpha, \beta$and γ are respectively, the first, second, and ith rows. By Lemma 2, $A[1, 2]$ must be one of the following four matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

which are the only nonsingular 2×2 symmetric 0–1 matrices. Let $A = (a_{ij})$. We have

$$a_{i1} = ua_{11} + va_{21}, \quad a_{i2} = ua_{12} + va_{22}, \tag{1}$$

$$a_{ii} = ua_{i1} + va_{i2} = u^2 a_{11} + 2uva_{12} + v^2 a_{22}. \tag{2}$$

We need to show $(u, v) \in \{(1, 0), (0, 1), (0, 0)\}$. Suppose this is not the case. Then

$$(u, v) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1), (0, -1), (-1, 0)\}.$$

(i)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

By (1), we must have $u = 1, \ v = -1$. But then by (2), $a_{ii} = -1,$ contradicting the fact that A is a 0–1 matrix.

(ii)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By (1), $u = v = 1$. But then by (2), $a_{ii} = 2,$ a contradiction.

(iii)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the same way as in Case (ii) we have a contradiction.

(iv)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

By (1), $u = -1, \ v = 1$. But then by (2), $a_{ii} = -1,$ a contradiction. The above contradictions complete the proof. ■

We remark that the conclusion of Lemma 3 does not hold when rank ≥ 3. Consider

$$E = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The first three rows of E are linearly independent, and the fourth row is the difference of the first two rows. So E is a symmetric 0–1 matrix of rank 3. But the fourth row is not equal to any of the first three rows and it is not a zero row. This example can be extended in an obvious way to a matrix of arbitrary rank $k \geq 3$ and of order $k+1$.

THEOREM 4 *Let k, n be positive integers with $k \leq n$. There exists an $n \times n$ symmetric* 0–1 *matrix of rank k with exactly d ones if and only if*

(i) $d = x^2$ *for some integer x with $1 \leq x \leq n$ when $k = 1$;*
(ii)

$$d = \begin{cases} s^2 - t^2 & \text{for some integers } s \text{ and } t \text{ with } 1 \leq t < s \leq n, \text{ or} \\ s^2 + t^2 & \text{for some integers } s \text{ and } t \text{ with } s \geq 1,\ t \geq 1 \text{ and } s + t \leq n, \text{ or} \\ 2st & \text{for some integers } s \text{ and } t \text{ with } s \geq 1,\ t \geq 1 \text{ and } s + t \leq n \end{cases}$$

when $k = 2$;
(iii) $k \leq d \leq n^2 - k + 1$ *when $k \geq 3$.*

Proof (i) Let $A = (a_{ij})$ be an $n \times n$ symmetric 0–1 matrix of rank 1. If all the diagonal entries of A are 0, then since A has at least one 1, A has a submatrix of the form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ which is nonsingular. Thus $\text{rank}(A) \geq 2$, contradicts the rank 1 assumption. So A has a diagonal entry, say, $a_{ii} = 1$. Suppose the ith row contains x 1's. Any other row is either a zero row or equal to the ith row. If $a_{ji} = 0$, then the jth row is a zero row; if $a_{ji} = 1$, then the jth row is equal to the ith row. But by the symmetry the ith column contains x 1's. Hence A has x rows equal to the ith row and has the remaining rows equal to 0. Therefore, A has x^2 1's.

Conversely, for $1 \leq x \leq n$ let J_x denote the all-one matrix of order x. Then the $n \times n$ matrix $\left[\begin{smallmatrix} J_x & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a symmetric 0–1 matrix of rank 1 and has x^2 1's.

(ii) Let $A = (a_{ij})$ be an $n \times n$ symmetric 0–1 matrix of rank 2. By simultaneous row and column permutations if necessary, we may suppose that the first two rows are linearly independent. By Lemma 2, $A[1, 2]$ is nonsingular. Thus $A[1, 2]$ is one of the following four matrices:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

If $A[1, 2] = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, then we may interchange the first two rows and interchange the first two columns so that $A[1, 2] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Therefore we need to consider only the first three possibilities. It is easy to check that the conclusions are true for $n = 2$. Next we assume $n \geq 3$.

Case 1 $A[1,2] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. By Lemma 3, each row of A is equal to the first row, or the second row, or 0. For every i, (a_{i1}, a_{i2}) is equal to $(1,1)$, or $(1,0)$, or $(0,0)$. Correspondingly the ith row is equal to the first row, or the second row, or 0. Suppose that the first row contains s 1's and that there are t rows equal to the second row. Using the symmetry of A and considering the numbers of 1's in the first and second columns, we see that $1 \le t < s \le n$, there are $s-t$ rows equal to the first row, and the second row contains $s-t$ 1's. The number of 1's in A is $(s-t)s + t(s-t) = s^2 - t^2$.

Denote by $J_{p,q}$ the $p \times q$ matrix all of whose entries are equal to 1, by $0_{p,q}$ the $p \times q$ zero matrix. We write J_p and 0_p for $J_{p,p}$ and $0_{p,p}$, respectively. Then for any $1 \le t < s \le n$, the matrix

$$G_1 = \begin{bmatrix} 1 & 1 & J_{1,s-t-1} & J_{1,t-1} & 0_{1,n-s} \\ 1 & 0 & J_{1,s-t-1} & 0_{1,t-1} & 0_{1,n-s} \\ J_{s-t-1,1} & J_{s-t-1,1} & J_{s-t-1} & J_{s-t-1,t-1} & 0_{s-t-1,n-s} \\ J_{t-1,1} & 0_{t-1,1} & J_{t-1,s-t-1} & 0_{t-1} & 0_{t-1,n-s} \\ 0_{n-s,1} & 0_{n-s,1} & 0_{n-s,s-t-1} & 0_{n-s,t-1} & 0_{n-s} \end{bmatrix}$$

is an $n \times n$ symmetric 0–1 matrix of rank 2 with $s^2 - t^2$ 1's.

Case 2 $A[1,2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Suppose the first row of A contains s 1's and the second row contains t 1's. Obviously $s \ge 1$, $t \ge 1$, and $s + t \le n$ by applying Lemma 3 to the columns of A. Using the same analysis as in Case 1, we deduce that the number of 1's in A is $s^2 + t^2$.

Conversely, for any $s \ge 1$, $t \ge 1$ with $s + t \le n$, the matrix

$$G_2 = \begin{bmatrix} 1 & 0 & J_{1,s-1} & 0_{1,t-1} & 0_{1,n-s-t} \\ 0 & 1 & 0_{1,s-1} & J_{1,t-1} & 0_{1,n-s-t} \\ J_{s-1,1} & 0_{s-1,1} & J_{s-1} & 0_{s-1,t-1} & 0_{s-1,n-s-t} \\ 0_{t-1,1} & J_{t-1,1} & 0_{t-1,s-1} & J_{t-1} & 0_{t-1,n-s-t} \\ 0_{n-s-t,1} & 0_{n-s-t,1} & 0_{n-s-t,s-1} & 0_{n-s-t,t-1} & 0_{n-s-t} \end{bmatrix}$$

is an $n \times n$ symmetric 0–1 matrix of rank 2 with $s^2 + t^2$ 1's.

Case 3 $A[1,2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Suppose the first row of A contains s 1's and the second row contains t 1's. Then $s \ge 1$, $t \ge 1$, $s + t \le n$. Using the same analysis once more as in Case 1, we deduce that the number of 1's in A is $2st$.

Conversely, for any $s \ge 1$, $t \ge 1$ with $s + t \le n$, the matrix

$$G_3 = \begin{bmatrix} 0 & 1 & J_{1,s-1} & 0_{1,t-1} & 0_{1,n-s-t} \\ 1 & 0 & 0_{1,s-1} & J_{1,t-1} & 0_{1,n-s-t} \\ J_{s-1,1} & 0_{s-1,1} & 0_{s-1} & J_{s-1,t-1} & 0_{s-1,n-s-t} \\ 0_{t-1,1} & J_{t-1,1} & J_{t-1,s-1} & 0_{t-1} & 0_{t-1,n-s-t} \\ 0_{n-s-t,1} & 0_{n-s-t,1} & 0_{n-s-t,s-1} & 0_{n-s-t,t-1} & 0_{n-s-t} \end{bmatrix}$$

is an $n \times n$ symmetric 0–1 matrix of rank 2 with $2st$ 1's.

(iii) For any rank $k \ge 1$, the number d of ones obviously satisfies $k \le d \le n^2 - k + 1$. Let I_k be the identity matrix of order k. Let $H_k = (h_{ij})$ be the $k \times k$ matrix with $h_{ii} = 0$ for $i = 2, 3, \ldots, n$ and with all other entries equal to 1. Then H_k is nonsingular. In fact,

$H_k^{-1} = [\begin{smallmatrix} 2-k & J_{1,k-1} \\ J_{k-1,1} & -I_{k-1} \end{smallmatrix}]$. So both $[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}]$ and $Z_{n,k} \equiv [\begin{smallmatrix} J_{n-k} & J_{n-k,k} \\ J_{k,n-k} & H_k \end{smallmatrix}]$ are of rank k and they have k and $n^2 - k + 1$ 1's, respectively. This shows that the lower bound k and the upper bound $n^2 - k + 1$ are attained. Now assume $k \geq 3$. Let $P(n, k)$ denote the proposition that for every positive integer d with $k \leq d \leq n^2 - k + 1$ there exists an $n \times n$ symmetric 0–1 matrix of rank k with exactly d ones. It remains to prove $P(n, k)$. We divide the proof into three steps.

Step 1 $P(n, 3)$ is true for all $n \geq 3$.

We use induction on n. The following matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

are of rank 3 and have $3, 4, 5, 6, 7$ ones, respectively. So $P(3, 3)$ holds. Denote by Ω_n^k the set of $n \times n$ symmetric 0–1 matrices of rank k. Suppose $P(n, 3)$ holds, i.e., there are matrices in Ω_n^3 with d ones for $3 \leq d \leq n^2 - 2$. We will show that $P(n + 1, 3)$ holds, i.e., there are matrices in Ω_{n+1}^3 with d ones for $3 \leq d \leq (n + 1)^2 - 2$. The range $3 \leq d \leq n^2 - 2$ is covered by $P(n,3)$: Just add one zero row and one zero column to the attaining matrices of order n. Consider

$$Z_{n,3} = \begin{bmatrix} J_{n-2} & J_{n-2,1} & J_{n-2,1} \\ J_{1,n-2} & 0 & 1 \\ J_{1,n-2} & 1 & 0 \end{bmatrix} \in \Omega_n^3.$$

Our strategy is to change the entries of the 2×2 submatrix in the right-bottom corner of $Z_{n,3}$ and add one row and one column to $Z_{n,3}$ so that the resulting matrix is in Ω_{n+1}^3 and has the required number of 1's. In the following matrices A_j, for each $j = 1, 2, 3, 4$, $A_j[n - 1, n, n + 1]$ is nonsingular and hence the last three rows of A_j are linearly independent, while every other row is a linear combination of these three rows. Thus $A_j \in \Omega_{n+1}^3$. Let

$$A_1 = \begin{bmatrix} J_{n-2} & J_{n-2,1} & J_{n-2,1} & 0_{n-2,1} \\ J_{1,n-2} & 1 & 0 & 0 \\ J_{1,n-2} & 0 & 0 & 1 \\ 0_{1,n-2} & 0 & 1 & 0 \end{bmatrix}.$$

The number of 1's in A_1 is $f(A_1) = n^2 - 1$. Let

$$A_2 = \begin{bmatrix} J_{n-2} & J_{n-2,1} & J_{n-2,1} & 0_{n-2,1} \\ J_{1,n-2} & 1 & 1 & 0 \\ J_{1,n-2} & 1 & 0 & 0 \\ 0_{1,n-2} & 0 & 0 & 1 \end{bmatrix}.$$

Then $f(A_2) = n^2$. Let

$$A_3 = \begin{bmatrix} J_{n-2} & J_{n-2,1} & J_{n-2,1} & 0_{n-2,1} \\ J_{1,n-2} & 1 & 1 & 0 \\ J_{1,n-2} & 1 & 0 & 1 \\ 0_{1,n-2} & 0 & 1 & 0 \end{bmatrix}.$$

Then $f(A_3) = n^2 + 1$. Let

$$A_4 = \begin{bmatrix} J_n & \alpha^T \\ \alpha & a_{n-1} \end{bmatrix}$$

where $\alpha = (a_1, a_2, \ldots, a_{n-2}, 1, 0)$ and $a_i = 0$ or 1 which will be specified. For any d with $n^2 + 2 \le d \le (n+1)^2 - 2$, if $d = n^2 + 2 + 2p$ for some nonnegative integer p, set $a_1 = a_2 = \cdots = a_p = 1$, $a_{p+1} = \cdots = a_{n-1} = 0$; if $d = n^2 + 2 + 2p + 1$ for some nonnegative integer p, set $a_1 = a_2 = \cdots = a_p = a_{n-1} = 1$, $a_{p+1} = \cdots = a_{n-2} = 0$. Then $f(A_4) = d$. Therefore $P(n+1, 3)$ holds.

Step 2 $P(n,k)$ implies $P(n+1, k+1)$.

Suppose $P(n,k)$ holds. Then for every d with $k \le d \le n^2 - k + 1$ there exists an $A_d \in \Omega_n^k$ with $f(A_d) = d$. We have $B_d \equiv \left[\begin{smallmatrix} A_d & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \Omega_{n+1}^{k+1}$ and $f(B_d) = d + 1$. So the range $k + 1 \le d \le n^2 - k + 2$ is attained by these B_d.

Denote by $Z_{n,k}$ the $n \times n$ 0–1 matrix whose only zero entries are the last $k-1$ diagonal entries. Our strategy is to construct matrices $W_j \in \Omega_{n+1}^{k+1}$ for $j = 1, 2, 3$ based on $Z_{n,k}$ with desired numbers d of 1's. For each j, the principal submatrix of A_j lying in the last $k+1$ rows is nonsingular and every other row is a linear combination of the last $k+1$ rows. Thus $W_j \in \Omega_{n+1}^{k+1}$. We will omit the verifications of these easily seen facts.

Let $W_1 = \left[\begin{smallmatrix} Z_{n,k} & \beta^T \\ \beta & 0 \end{smallmatrix}\right]$ where $\beta = (0, 0, \ldots, 0, 1)$. Then $f(W_1) = n^2 - k + 3$.

Let $W_2 = \left[\begin{smallmatrix} Z_{n,k} & \beta^T \\ \beta & 1 \end{smallmatrix}\right]$ where β is as mentioned earlier. Then $f(W_2) = n^2 - k + 4$.

Let

$$W_3 = \begin{bmatrix} J_{n-k+1} & J_{n-k+1,k-1} & \gamma^T \\ J_{k-1,n-k+1} & H_{k-1} & \omega^T \\ \gamma & \omega & a_{n-1} \end{bmatrix}$$

where H_{k-1} is the 0–1 matrix of order $k-1$ whose only zero entries are the last $k-2$ diagonal entries, $\gamma = (a_1, \ldots, a_{n-k}, 1)$, $\omega = (0, a_{n-k+1}, \ldots, a_{n-2})$, and the a_i are to be specified. For any d with $n^2 - k + 5 \le d \le (n+1)^2 - (k+1) + 1$, if $d = n^2 - k + 4 + 2p$, set $a_1 = \cdots = a_p = 1$, $a_{p+1} = \cdots = a_{n-1} = 0$; if $d = n^2 - k + 4 + 2p + 1$, set $a_1 = \cdots = a_p = a_{n-1} = 1$, $a_{p+1} = \cdots = a_{n-2} = 0$. Then $f(W_3) = d$. Thus we have proved $P(n+1, k+1)$.

Step 3 $P(n,k)$ is true for all $n \ge k \ge 3$.

By Step 1, $P(n-k+3, 3)$ is true. Using Step 2 we have the following implications:

$$P(n-k+3, 3) \Rightarrow P(n-k+4, 4) \Rightarrow \cdots \Rightarrow P(n,k).$$

This completes the proof. ∎

3. Examples

Example 1 By Theorem 1 (i),

$$\{1, 2, 3, 4, 6, 8, 9, 12, 16\}$$
$$= \{d \mid \text{There is a } 4 \times 4 \ 0{-}1 \text{ matrix of rank 1 with } d \text{ 1's}\}.$$

Note that $5, 7, 10, 11, 13, 14, 15$ are missing. By Theorem 4 (i),

$$\{1, 4, 9, 16\}$$
$$= \{d \mid \text{There is a } 4 \times 4 \text{ symmetric } 0{-}1 \text{ matrix of rank 1 with } d \text{ 1's}\}.$$

Example 2 By Theorem 4 (ii),

$$\{2, 3, 4, 5, 6, 7, 8, 10, 12, 15\}$$
$$= \{d \mid \text{There is a } 4 \times 4 \text{ symmetric } 0{-}1 \text{ matrix of rank 2 with } d \text{ 1's}\}.$$

Note that $9, 11, 13, 14$ are missing.

Example 3 By Theorem 4 (iii),

$$\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$$
$$= \{d \mid \text{There is a } 4 \times 4 \text{ symmetric } 0{-}1 \text{ matrix of rank 3 with } d \text{ 1's}\}.$$

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