Standard Domineering

Sahil Bahri
Mathematics and Computing
IIT Delhi
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Clyde P. Kruskal
Department of Computer Science
University of Maryland, College Park

Abstract

Cincotti and Iida invented the game of of Synchronized Domineering, and analyzed a few special cases. We developed a more general technique of analysis, and obtained results for many more special cases. This article lists application of our analysis to obtain results for standard two player Domineering

Introduction

In Synchronized Domineering, exist four possible outcomes:

- G=H
- G=V
- G=1st
- G=2nd

We apply a similar anaysis as applied to Synchronized Domineering.

Notation to account for the number of moves made

Same as before

Notation to account for the number of moves reserved

Same as before

Theorem 0.1 *Let 'x' be the number of saturated states for subboards of size m×2 of a rectangle of Standard Domineering of size m×n.*

Consider any combination of the 'x' states for whom $\sum_{k=1}^{\lceil n/2 \rceil} N_i = 0$
Assume for such a combination $\sum_{k=1}^{\lceil n/2 \rceil} H_i = H_a$
$\sum_{k=1}^{\lceil n/2 \rceil} h_i = h_a$
$\sum_{k=1}^{\lceil n/2 \rceil} V_i = V_a$
$\sum_{k=1}^{\lceil n/2 \rceil} v_i = v_a$
(n mod 2) $\times \lfloor m/2 \rfloor =$ *freemoves*

Consider any combination of the 'x' states for whom $\sum_{k=1}^{\lceil n/2 \rceil} N_i = 1$
Assume for such a combination $\sum_{k=1}^{\lceil n/2 \rceil} H_i = H_b$
$\sum_{k=1}^{\lceil n/2 \rceil} h_i = h_b$
$\sum_{k=1}^{\lceil n/2 \rceil} V_i = V_b$
$\sum_{k=1}^{\lceil n/2 \rceil} v_i = v_b$
(n mod 2) $\times \lfloor m/2 \rfloor =$ *freemoves*

Consider any combination of the 'x' states for whom $\sum_{k=1}^{\lceil n/2 \rceil} N_i = -1$
Assume for such a combination $\sum_{k=1}^{\lceil n/2 \rceil} H_i = H_c$
$\sum_{k=1}^{\lceil n/2 \rceil} h_i = h_c$
$\sum_{k=1}^{\lceil n/2 \rceil} V_i = V_c$
$\sum_{k=1}^{\lceil n/2 \rceil} v_i = v_c$
(n mod 2) $\times \lfloor m/2 \rfloor =$ *freemoves*

- *G=H or 1st iff*

$$H_a + \lceil h_a/2 \rceil > V_a + \lceil v_a/2 \rceil + freemoves \quad (1)$$

and

$$H_b + \lceil h_b/2 \rceil \geq V_b + \lceil v_b/2 \rceil + freemoves \quad (2)$$

- *G=H or 2nd iff*

$$H_a + \lceil h_a/2 \rceil \geq V_a + \lceil v_a/2 \rceil + freemoves \quad (3)$$

and

$$H_c + \lceil h_c/2 \rceil > V_c + \lceil v_c/2 \rceil + freemoves \quad (4)$$

G=H if G=H or 1st and G=H or 2nd

Three possible combinations of board are worth considering

$$\sum_{k=1}^{\lceil n/2 \rceil} N_i = 0 \quad (5)$$

Equation 5 illustrates the case when both players have made the same number of moves.

$$\sum_{k=1}^{\lceil n/2 \rceil} N_i = 1 \tag{6}$$

Equation 6 illustrates the case when Harvey has made an extra move before saturation across all boards. This implies that Harvey went in first and Vicky couldn't follow him into a valid spot (Reserved Spot or overlapping him incase of Synchronized Domineering).

$$\sum_{k=1}^{\lceil n/2 \rceil} N_i = -1 \tag{7}$$

Equation 7 illustrates the case when Vicky has made an extra move before saturation across all boards. This implies that Vickey went in first and Harvey couldn't follow him into a valid spot (Reserved Spot or overlapping him incase of Synchronized Domineering).

The difference between the number of moves made by both players can at no point in time exceed 1. We eliminate all other possibilites because of the alternating pattern in which both players make their move.

We will justify elimination of equation 2 for proofs of Synchronized Domineering. The definition of saturated states implies that any move that Harvey makes has a positive value for H or v. This is to say that he reserves a spot (half or full or both) for himself. Vicky has the power to move over Harvey's move hence always has a legal move into the board at this stage of the game. Hence, in the Synchronized Version we only considered combinations for which $\sum_{k=1}^{\lceil n/2 \rceil} N_i = 0$

For the standard Version however, their is no such guarantee and we consider all three possible terminating conditions.

Note that equation 1 is just a stricter form of equation 3. To prove G=H, we hence need to satisfy equations 1, 2 and 4. Note that the set of saturated boards for Synchronized Domineering are a superset of the set of saturated boards for Standard Domineering for a given dimension. This is so because Synchronized Domineering saturated states are nothing, but saturated states of Standard Domineering plus staurated states obtained by incorporationg possibility of overlap. Since equation 1 was shown to be satisfied for the saturated version of domineering for certain dimensions in theorem 3, 5 and 7, it will also fold for standard Domineering played on the same dimension of the board. We fail to get any results for the even $\times$ n (sufficiently large) board due to violation of equation 1 and equation 4.

In the following proofs, we assume that the strategy adopted by Harvey is the same strategy that he adopted in the proofs for the same dimensions of Synchronized Domineering.

Theorem 0.2 *Let G=3×n* $\{\forall n \in \mathbf{N} : n \geq 4\}$ *be a rectangle of Domineering. Then G=H*

Note that equation 1 holds (by the argument given above). Also State 2 in the Table 5 of Synchronized Domineering is not valid here. Equations 3 and 4 are satisfied for all values of n≥4 Note that state 6 independenty does not satisfy equation 4. However, when combined with other states, it will have to be combined with atleast one state that has N $\geq$ 0. This leads to satisfaction of equation 4 with a lower bound of n=4 $\blacksquare$

Theorem 0.3 *Let G=7×n* $\{\forall n \in \mathbf{2N} : n \geq 4\}$ $\{\forall n \in \mathbf{2N+1} : n \geq 33\}$ *be a rectangle of Domineering. Then G=H*

Note that equation 1 holds (by the argument given above).

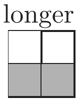
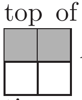
Some states of Synchronized Domineering are no longer valid.

Equations 3 and 4 are satisfied for all even n greater than equal to 4. Despite the fact that each of the states in Table 7 of synchronized Domineering do not independly satisfy equation 4, when combined with other states, such states will have to be combined with atleast one state that has N $\geq$ 0. This leads to satisfaction of equation 4 with a lower bound of n=4

In the synchronized version of the game for odd n, we had justified a lower bound of 33 for satisfying eqaution 1. The reason was that since freemoves = 3, Harvey needed to exceed Vicky's Reserved Moves by 4. Since N = -1 in the states in table 7, and these are the only states that give Harvey no advantage, such states are beneficial to Vicky, and the worst

case is when 3 of such states are compensated by 1 state with N=3. (Since N ≤ 3 on a 7 by 2 saturated board). So for every gained move we had four saturated boards. Since four moves had to be gained the lower bound is the least odd number greater than 4 times the combined width of four boards. This is equal to 33.
Equations 3 and 4 however, will be satisfied with a lower bound on odd n. Hence an overall odd bound of 33 satisfies all the required equations. $\blacksquare$

Theorem 0.4 *Let G=9×n* $\{\forall n \in \mathbf{2N} : n \geq 2\}$ $\{\forall n \in \mathbf{2N+1} : n \geq 21\}$ *be a rectangle of Domineering. Then G=H*

Note that equation 1 holds (by the argument given above).
Some states of Synchronized Domineering for 7×2 are no longer valid. We shall obtain new states by adding  (increases N by 1 and H by 1)to the top of states of the previous proof or by adding  to the bottom of the states under consideration.
Equations 1, 3 and 4 will be satisfied for the stated bounds. $\blacksquare$