Spring 2010 **CMSC 858Y: Homework 1** Samir Khuller

Due in class: No official due date. Complete it before Mar 1..

(1) A *maximal matching* is a matching M such that no edges can be added to the matching, without destroying the property that it is a matching. In other words, for any edge e, $e \cup M$ is not a matching. First give a linear time algorithm for finding a maximal matching in a graph. Then prove that the size of the maximal matching is at least half the size of a maximum matching.

(2) Consider a set of costs $c_{ij} \geq 0$ on the edges of a complete graph K_n with n nodes, and let $S \subseteq \{v_1, \ldots, v_n\}$. We wish to find a subgraph of K_n that has odd degree at nodes in S, an even degree (possibly zero) at all other nodes, and as low cost as possible. Show that this is a minimum weight matching problem. (Min weight matching can be solved in polynomial time in graphs that are not necessarily bipartite.)

(3) We are given a connected graph $G = (V, E)$ and a cost function $c : E \rightarrow Z^+$. We wish to find a walk, traversing each edge of E at least once, such that the total cost of the walk (with multiple traversals of an edge charged multiply) is as small as possible. Give a polynomial time algorithm for this problem. (You may assume a solution to problem (2) even if you did not solve problem (2).)

(4) The $b-$*matching* problem is the following: given a bipartite graph $B = (V, U, E)$ and a function $b : V \cup U \rightarrow Z^+$, is there a subset $M \subseteq E$ such that for each $v \in V \cup U$, v is incident upon exactly $b(v)$ edges in M? How would one design a strongly polynomial algorithm for this problem? (An algorithm is strongly polynomial if its running time depends only on the input size (i.e., number of vertices or edges) and not on the particular numbers/weights. In this case the running time should not depend on the b values.)

(5) Let $G = (X, Y, E)$ be a bipartite graph. Assume for simplicity that $|X| = |Y|$.

A subset $S \subseteq (X \cup Y)$ is called a *vertex cover* if for *each* edge $e = (x, y)$ in E either $x \in S$ or $y \in S$ (or both).

Let M^* be a maximum matching in G.

(a) Prove that if S is a vertex cover then $|S| \geq |M^*|$.

(b) Give a polynomial time algorithm to find the minimum cardinality vertex cover. Give a proof for its correctness.

(6) Design a polynomial time algorithm for the following problem. You have a directed network, with two special vertices s and t. Each edge e has a positive capacity $c(e)$ and a positive delay $d(e)$. The time to ship σ units of data from s to t on path P is $\sum_{e \in P} d(e) + \frac{\sigma}{C}$ where $C = \min_{e \in P} c(e)$. Design a polynomial time algorithm to find the path that will minimize the shipping time? You should also give an argument that shows why your method works.