Notes by Samir.

4 Linear Programming

Linear programming is one of the most general problems known to be solvable in polynomial time. Many optimization problems can be cast directly as polynomial-size linear programs and thus solved in polynomial time. Often the theory underlying linear programming is useful in understanding the structure of optimization problems. For this reason, L.P. theory is often useful in *designing* efficient algorithms for an optimization problem and for understanding such algorithms within a general framework.

Here is an example of a linear programming problem.

$$\begin{array}{ll} \max & (5x_1 + 4x_2 + 3x_3) \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 \le 5 \\ & 4x_1 + x_2 + 2x_3 \le 11 \\ & 3x_1 + 4x_2 + 2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

The goal is to find values (real numbers) for the x_i's meeting the constraints so that the *objective function* is maximized. Note that all of the constraints are linear inequalities or equations and the objective function is also linear. In general, a *linear program* is a problem of maximizing or minimizing a linear function of some variables subject to linear constraints on those variables.

Simplex Method. The most common method of solving linear programs is the *simplex algorithm*, due to Dantzig. The method first converts the linear inequalities into equality constraints by introducing *slack* variables. The original variables are referred to as the *decision* variables.

$$\begin{array}{ll} \text{define} & \\ & x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ & x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ & x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & z = 5x_1 + 4x_2 + 3x_3 \\ & \quad x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \\ \max & z \end{array}$$

It then starts with an initial *feasible solution* (an assignment to the variables meeting the constraints, but not necessarily optimizing the objective function). For instance, let $x_1, x_2, x_3 = 0$, so that $x_4 = 5, x_5 = 11, x_6 = 8$ gives a feasible solution. The simplex method then does successive improvements, changing the solution and increasing the value of z, until the solution is optimal.

Clearly, if we can increase x_1 without changing x_2 or x_3, we can increase the value of z. We consider increasing x_1 and leaving x_2 and x_3 zero, while letting the equations for x_4, x_5, and x_6 determine their values. If we do this, how much can we increase x_1 before one of x_4, x_5 or x_6 becomes zero? Clearly, since $x_4 \ge 0$, we require that $x_1 \le \frac{5}{2}$. Since $x_5 \ge 0$, we require that $x_1 \le \frac{11}{4}$. Similarly, since $x_6 \ge 0$, we require that $x_1 \le \frac{8}{3}$. We thus increase x_1 to $\frac{5}{2}$, and see that $z = 12.5$. Now $x_2, x_3, x_4 = 0$ and x_1, x_5, x_6 are non-zero.

This operation is called a *pivot*. A pivot improves the value of the objective function, raises one zero variable, brings one non-zero variable down to zero.

How do we continue ? We rewrite the equations so that all the non-zero variables (simplex calls these the "basic" variables) are expressed in terms of the zero variables. In this case, this yields the following equations.

$$\begin{array}{ll} \max & z \\ \text{s.t.} & x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{array}$$

$$
\begin{aligned}
x_5 &= 1 + 5x_2 + 2x_4 \\
x_6 &= \tfrac{1}{2} + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4 \\
z &= \tfrac{25}{2} - \tfrac{7}{2}x_2 + \tfrac{1}{2}x_3 - \tfrac{5}{2}x_4 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

The natural choice for the variable to increase is x_3. It turns out that we can increase x_3 to 1, and then x_6 becomes 0. Then, the new modified system we get (by moving x_3 to the LHS) is:

$$
\begin{aligned}
\text{max} \quad & z \\
\text{s.t.} \quad & x_3 = 1 + x_2 + 3x_4 - 2x_6 \\
& x_1 = 2 - 2x_2 - 2x_4 + x_6 \\
& x_5 = 1 + 5x_2 + 2x_4 \\
& z = 13 - 3x_2 - x_4 - x_6 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Now we have $z = 13$ (with the current solution in which $x_2, x_4, x_6 = 0$). Also notice that increasing any of the zero variables will only decrease the value of z. Since this system of equations is equivalent to our original system, we have an optimal solution.

Possible Problems:

Initialization: Is it easy to get a starting feasible solution ?

Progress in an iteration: Can we always find a variable to increase if we are not at optimality?

Termination: Why does the algorithm terminate ?

Their are fairly easy ways to overcome each of these problems.

Linear Programming Duality

Given an abstract linear programming problem (with n variables and m constraints), we can write it as:

$$\begin{array}{lll} \max & z = c^T x & c^T = (1 \times n),\ x = (n \times 1) \\ \text{s.t.} & Ax \leq b & A = (m \times n),\ b = (m \times 1) \\ & x \geq 0 & \end{array}$$

Simplex obtains an optimal solution (i.e., maximizes z). Notice that *any feasible* solution to the `primal` problem yields a lower bound on the value of z. The entire motivation for studying the linear programming `dual` is a quest for an *upper* bound on the value of z.

Consider the following linear programming problem:

$$\begin{array}{ll} \max & z = 4x_1 + x_2 + 5x_3 + 3x_4 \\ \text{s.t.} & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\ & 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\ & -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Notice that $(0,0,1,0)$ is a feasible solution to the primal, and this yields the value of $z = 5$, implying that $z^* \geq 5$ (z^* is the optimum value). The solution $(3,0,2,0)$ is also feasible and shows that $z^* \geq 22$.

If we had a solution, how would we know that it was optimal ?

Here is one way of giving an upper bound on the value of z^*. Add up equations (2) and (3). This gives us $4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$. Clearly, for non-negative values of x_i this is always $\geq z$, $(4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58)$. This gives us an upper bound on the value of z^* of 58.

In general, we are searching for a linear combination of the constraints to give an upper bound for the value of z^*. What is the best combination of constraints that will give us the smallest upper bound ? We will formulate this as a linear program ! This will be the dual linear program.

Consider multiplying equation (i) $(i = 1,2,3)$ in the above example by y_i and adding them all up. We obtain

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3$$

We also require that all $y_i \geq 0$.

We would like to use this as an upper bound for z. Since we would like to obtain the tightest possible bounds (want to minimize the RHS), we could rewrite it as the following LP.

$$\begin{array}{l} y_1 + 5y_2 - y_3 \geq 4 \\ -y_1 + y_2 + 2y_3 \geq 1 \\ -y_1 + 3y_2 + 3y_3 \geq 5 \\ 3y_1 + 8y_2 - 5y_3 \geq 3 \\ y_1, y_2, y_3 \geq 0 \end{array}$$

If these constraints are satisfied, we conclude that $z \leq y_1 + 55y_2 + 3y_3$. If we try to minimize $(y_1 + 55y_2 + 3y_3)$ subject to the above constraints, we obtain the *dual linear program.*

The dual linear program can be written succinctly as:

$$\begin{array}{lll} \min & b^T y & b^T = (1 \times m),\ y = (m \times 1) \\ \text{s.t.} & y^T A \geq c^T & A = (m \times n),\ c^T = (1 \times n) \\ & y \geq 0 & \end{array}$$

Primal LP (n variables, m equations):

$$\begin{array}{lll} & \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1 \ldots m) \\ & x_j \geq 0 & (j = 1 \ldots n) \end{array}$$

Dual LP (m variables, n equations):

$$\begin{array}{lll} & \min \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & (j = 1 \ldots n) \\ & y_i \geq 0 & (i = 1 \ldots m) \end{array}$$

Theorem 4.1 (Weak Duality Theorem)
For every primal feasible solution x, and every dual feasible solution y, we have:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i.$$

Proof:
The proof of this theorem is really easy and follows almost by definition.

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \leq \sum_{i=1}^{m} b_i y_i$$

$\square$

It is easy to see that if we obtain x^* and y^*, such that the equation in the Weak Duality theorem is met with equality, then both the solutions, x^*, y^* are optimal solutions for the primal and dual programs. By the Weak Duality theorem, we know that for any solution x, the following is true:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i^* = \sum_{j=1}^{n} c_j x_j^*$$

Hence x^* is an optimal solution for the primal LP. Similarly, we can show that y^* is an optimal solution to the Dual LP.

Theorem 4.2 (Strong Duality Theorem)
If the primal LP has an optimal solution x^*, then the dual has an optimal solution y^* such that:

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

Proof:
To prove the theorem, we only need to find a (feasible) solution y^* that satisfies the constraints of the Dual LP, and satisfies the above equation with equality. We solve the primal program by the simplex method, and introduce m slack variables in the process.

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad (i = 1, \ldots, m)$$

Assume that when the simplex algorithm terminates, the equation defining z reads as:

$$z = z^* + \sum_{k=1}^{n+m} \overline{c}_k x_k.$$

Since we have reached optimality, we know that each $\overline{c}_k$ is a nonpositive number (in fact, it is 0 for each basic variable). In addition z^* is the value of the objective function at optimality, hence $z^* = \sum_{j=1}^{n} c_j x_j^*$. To produce y^* we pull a rabbit out of a hat ! Define $y_i^* = -\overline{c}_{n+i} \quad (i = 1, \ldots, m)$. To show that y^* is an optimal dual feasible solution, we first show that it is feasible for the Dual LP, and then establish the strong duality condition.

¿From the equation for z we have:

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{k=1}^{n} \overline{c}_k x_k - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j).$$

Rewriting it, we get

$$\sum_{j=1}^{n} c_j x_j = (z^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (\overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*) x_j.$$

Since this holds for all values of x_i, we obtain:

$$z^* = \sum_{i=1}^{m} b_i y_i^*$$

(this *establishes the equality*) and

$$c_j = \overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \quad (j = 1, \ldots, n).$$

Since $\overline{c}_k \leq 0$, we have

$$y_i^* \geq 0 \quad (i = 1, \ldots, m).$$

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \quad (j = 1, \ldots, n)$$

This *establishes the feasibility* of y^*. □

Complementary Slackness Conditions:

Theorem 4.3 *Necessary and Sufficient conditions for x^* and y^* to be optimal solutions to the primal and dual are as follows.*

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ or } x_j^* = 0 \text{ (or both) for } j = 1, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \text{ or } y_i^* = 0 \text{ (or both) for } i = 1, \ldots, m$$

In other words, if a variable is non-zero then the corresponding equation in the dual is met with equality, and vice versa.

Proof:

We know that

$$c_j x_j^* \leq (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^* \qquad (j = 1, \ldots, n)$$

$$(\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* \leq b_i y_i^* \qquad (i = 1, \ldots, m)$$

We know that at optimality, the equations are met with equality. Thus for any value of j, either $x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$. Similarly, for any value of i, either $y_i^* = 0$ or $\sum_{j=1}^{n} a_{ij} x_j^* = b_i$. □