CMSC 858S: Randomized Algorithms

Fall 2001

Handout 6: The vertex-connectivity of random graphs

In this short handout, we complete the discussion from class on the vertex-connectivity of random graphs. As mentioned in class, stronger results are known; see [1]. The discussion here is adapted from [2].

Recall the context. We choose a random graph G from $G(n,p)$, where $p = p(n) \geq (\ln^2 n)/n$, say. Fix any constant $\delta > 0$, and let $k = \lceil (n-1)p(1-2\delta) \rceil$. We wish to prove that as $n \to \infty$, the probability that G is *not* k-vertex-connected, tends to 0. We now zoom to the point at which we stopped this argument in class. Let $t = \lceil (n-1)p\delta \rceil$, and $s = k-1$. We aim to show that

$$q = \binom{n}{s} \cdot \sum_{i=t}^{(n-s)/2} \binom{n-s}{i} \cdot (1-p)^{i(n-s-i)}$$

tends to 0 as $n \to \infty$, under the above-seen assumptions on p, δ etc.

We start with some simplifications. We have

$$\begin{aligned}
q &= \binom{n}{\min\{s, n-s\}} \cdot \sum_{i=t}^{(n-s)/2} \binom{n-s}{i} \cdot (1-p)^{i(n-s-i)} \\
&\leq n^{\min\{s,n-s\}} \cdot \sum_{i=t}^{(n-s)/2} \binom{n-s}{i} \cdot (1-p)^{i(n-s-i)} \\
&\leq n^{\min\{s,n-s\}} \cdot \sum_{i=t}^{(n-s)/2} \binom{n-s}{i} \cdot e^{-ip(n-s-i)} \\
&\leq n^{\min\{s,n-s\}} \cdot \sum_{i=t}^{(n-s)/2} \binom{n-s}{i} \cdot e^{-ip(n-s)/2} \quad \text{(since } i \leq (n-s)/2\text{)} \\
&\leq n^{\min\{s,n-s\}} \cdot \sum_{i=t}^{(n-s)/2} (n-s)^i \cdot e^{-ip(n-s)/2} \\
&= n^{\min\{s,n-s\}} \cdot \sum_{i=t}^{(n-s)/2} e^{i(\ln(n-s) - p(n-s)/2)}. \qquad (1)
\end{aligned}$$

We now consider two cases. First suppose $s \leq n/2$. Since $n - s \geq n/2$, we get from (1), if n is large enough, that

$$\begin{aligned}
q &\leq n^s \cdot \sum_{i=t}^{(n-s)/2} e^{i(\ln n - np/4)} \\
&\leq n^s \cdot \sum_{i=t}^{(n-s)/2} e^{-\Omega(inp)} \quad \text{(since } np \gg \ln n\text{)} \\
&\leq e^{O(np \ln n) - \Omega(npt)} \\
&\to 0,
\end{aligned}$$

since $t \gg \ln n$ for any fixed $\delta > 0$, as $n \to \infty$.

Next suppose $s > n/2$. Here, we must have $p \geq 1/2$; so, (1) shows that

$$q \leq n^{n-s} \cdot \sum_{i=t}^{(n-s)/2} e^{i(\ln(n-s)-(n-s)/4)}.$$

Note that $n - s \geq 2\delta n$. So, since δ is some positive constant, we have when n gets arbitrarily large that $\ln(n-s) \ll (n-s)/4$. Thus,

$$\begin{aligned} q &\leq n^{n-s} \cdot \sum_{i=t}^{(n-s)/2} e^{-\Omega(i(n-s))} \\ &\leq e^{O((n-s)\ln n)-\Omega(t(n-s))} \\ &\rightarrow 0, \end{aligned}$$

again since $t \gg \ln n$ and since $(n-s)$ is arbitrarily large as $n \rightarrow \infty$.

References

[1] B. Bollobás. *Random Graphs.* Academic Press, London, 1985.

[2] A. Srinivasan, K. G. Ramakrishnan, K. Kumaran, M. Aravamudan, and S. Naqvi. Optimal Design of Signaling Networks for Internet Telephony. In *Proc. IEEE Conference on Computer Communications*, pages 707–716, 2000.