CMSC 858S: Randomized Algorithms

Fall 2001

Mid-Term Exam

Instructions: There are three questions, arranged approximately in increasing order of difficulty. If you think you have some ideas that deserve partial credit, please *itemize them concisely.* Also, if you are unable to solve a problem but can do so by making some assumptions, please clearly state these assumptions and then proceed. There are 20 points in total. Good luck!

1. (5 **points**) We consider here an extension of the "edge-disjoint paths with paths specified" problem that we studied in class. The setting is the same as in class, **except** that each edge must have at most 2 paths using it. (The case studied in class was where each edge must have at most 1 path using it. You can assume the same notation that we used in class, such as the graph $G = (V, E)$, the pairs of vertices $\{(s_i, t_i) : 1 \leq i \leq k\}$, and the collection of paths $\mathcal{P}_i$ for each (s_i, t_i)-pair.) Develop an approximation algorithm for this problem, and prove what its approximation guarantee is.

2. (5 **points**) We are given n (not necessarily independent) random variables $X_1, X_2, \ldots, X_n$, where each X_i takes on values only in $\{0, 1\}$. Let $X = X_1 + X_2 + \cdots + X_n$, and let $a \leq n$ be a positive integer. Show that for every positive integer $k \leq a$, we have

$$\Pr[X \geq a] \leq \frac{\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathbf{E}[X_{i_1} X_{i_2} \cdots X_{i_k}]}{\binom{a}{k}}. \tag{1}$$

The sum in inequality (1) is taken over all tuples of k integers $(i_1, i_2, \ldots, i_k)$ which satisfy $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Also, $\binom{a}{k}$ as usual denotes "a choose k", i.e., $a!/(k!(a-k)!)$.

3. We are given an undirected graph $G = (V, E)$. Let $\lambda \in [1, 2]$. Suppose we are given a value x_i for each vertex i and a value $z_{i,j}$ for each edge (i, j), such that:

(a) $0 \leq x_i \leq 1/\lambda$ for each i, and $z_{i,j} \in [0, 1]$ for each edge (i, j); and

(b) for each edge (i, j), we have $x_i + x_j \geq z_{i,j}$.

Consider the following random process: independently for each vertex i, set a random variable X_i to 1 with probability λx_i, and to 0 with the remaining probability of $1 - \lambda x_i$. Let Y be the random variable denoting the number of edges (i, j) for which at least one of X_i and X_j is one. We want to claim that with good probability, Y is "large enough".

(i) (3 **points**) Show that $\mathbf{E}[Y] \geq (\lambda - \lambda^2/4) \cdot (\sum_{(i,j):\ (i,j) \in E} z_{i,j})$.

(ii) Suppose every vertex in G has at most d edges incident upon it: i.e., G has maximum degree d. Also, suppose $\mathbf{E}[Y] = \alpha$. Find as good an upper-bound as you can for $\Pr[Y \leq \alpha - 5\sqrt{d\alpha}]$, in each of the following two cases:

- (2 **points**) the case where $d = 1$, and
- (5 **points**) the case where d is some arbitrary given number.
  [**Hint:** Which of the tail bounds that we have discussed, can be useful here?]