CMSC 858S: Randomized Algorithms

Fall 2001

Ungraded Homework Assignment #3

The suggested time period to finish this ungraded assignment is 10 days.

The first three problems are related to the Lovász Local Lemma (LLL); let us first discuss some notation related to the LLL that will be useful in solving these problems. Suppose we have some "bad" events $E_1, E_2, \ldots, E_m$, and want to show that $\Pr[\bigwedge_{i=1}^m \overline{E_i}] > 0$. This notation of bad events E_i will be used in the first three problems below. As in the LLL, we want to define for each $i, 1 \leq i \leq m$, a subset $S_i \subseteq (\{1, 2, \ldots, m\} - \{i\})$ of "small" cardinality such that S_i is the "dependent set" of event E_i. Convince yourself, using the proof of the LLL, that the following condition is sufficient for S_i to indeed be the dependent set of E_i:

$$\forall i \ \forall A_i \subseteq \overline{(S_i \cup \{i\})}, \quad \Pr[E_i \mid \bigwedge_{j \in A_i} \overline{E_j}] = \Pr[E_i]. \tag{1}$$

Next, again using the proof of the LLL, note that instead of (1), the following weaker requirement is in fact sufficient for S_i to be the dependent set of E_i:

$$\forall i \ \forall A_i \subseteq \overline{(S_i \cup \{i\})}, \quad \Pr[E_i \mid \bigwedge_{j \in A_i} \overline{E_j}] \leq \Pr[E_i]. \tag{2}$$

This weaker requirement is sometimes called the *Lopsided Local Lemma*, and we will see its use in problem 3 below.

1. In this problem, we study an abstract setting where we can rigorously prove that for each i, a certain set S_i is indeed the dependent set of E_i. The amount of notation in this problem is somewhat large, so you may need one or two repeated readings to digest all the notation. However, most of the notation is simple.

Suppose we have some *independent* random variables $X_1, X_2, \ldots, X_n$, each taking values in some finite set. Suppose we have subsets $B_1, B_2, \ldots, B_m$ of $\{1, 2, \ldots, n\}$ such that each E_i is completely determined by the values of $(X_j : j \in B_i)$. Then, since the X_i are all independent, it seems intuitively clear that the dependent set S_i of E_i, can be taken to be

$$S_i \doteq \{j : \ j \neq i, \text{ and } B_j \cap B_i \neq \emptyset\}. \tag{3}$$

Show that this definition of S_i indeed satisfies (1).

Hint: One approach to doing this is to start as follows. Let Z_i denote the tuple of all the random variables X_j *except for* the random variables $(X_k : k \in B_i)$. [As an example, suppose $n = 5$ and $B_i = \{2, 4\}$. Then, Z_i could be the tuple (X_1, X_3, X_5).] Then, if B_i has cardinality s, and if each random variable X_j takes values in some set C, note that Z_i takes on values in the set C^{n-s}, which is the Cartesian product of the set C with itself, taken $n - s$ times. We need to prove that the definition (3) satisfies (1), for an arbitrary $A_i \subseteq \overline{(S_i \cup \{i\})}$. To do so, start with the observation

$$\Pr[E_i \mid \bigwedge_{j \in A_i} \overline{E_j}] = \sum_{\sigma \in C^{n-s}} (\Pr[(Z_i = \sigma) \mid \bigwedge_{j \in A_i} \overline{E_j}] \cdot \Pr[E_i \mid ((Z_i = \sigma) \wedge \bigwedge_{j \in A_i} \overline{E_j})]);$$

in the right-hand-side, we adopt the convention that if $\mathcal{E}$ is a zero-probability event, then $\Pr[\mathcal{F} \mid \mathcal{E}]$ is taken to be zero. Now complete the proof.

The objective of this problem is to show a rigorous method to define dependent sets in general. Now go back to the packet-routing application seen in class, and verify that the definition of dependent sets that we used there is correct.

2. The version of the LLL presented in class is called the *symmetric* version. There is a stronger asymmetric version that also holds, which we now show. First note that

$$\Pr[(\bigwedge_{i=1}^{d} \overline{B_i}) \mid C] = \prod_{i=1}^{d} \Pr[\overline{B_i} \mid (C \wedge \bigwedge_{j<i} \overline{B_j})], \tag{4}$$

and recall that we used the lower bound $\Pr[(\bigwedge_{i=1}^{d} \overline{B_i}) \mid C] \geq 1 - \sum_{i=1}^{d} \Pr[B_i \mid C]$ in class.

(a) Now show the following:

LLL (Asymmetric version). *Suppose we have events $E_1, E_2, \ldots, E_m$. Suppose further that for each i, we can define: (a) a set $S_i \subseteq (\{1, 2, \ldots, m\} - \{i\})$ for which (2) holds, and (b) a real value $0 \leq x_i < 1$, such that*

$$\forall i, \quad \Pr[E_i] \leq x_i \cdot \prod_{j \in S_i} (1 - x_j). \tag{5}$$

Then, $\Pr[\bigwedge_{i=1}^{m} \overline{E_i}] \geq \prod_{i=1}^{m} (1 - x_i) > 0$.

Hint: Instead of the induction hypothesis seen in class, try showing, for all S that does not contain i, that $\Pr[E_i \mid \bigwedge_{j \in S} \overline{E_j}] \leq x_i$. Again use induction on $|S|$, and also use (4).

(b) The LLL version seen in class says that if all E_i have $\Pr[E_i] \leq p$ and have $|S_i| \leq D$, then it suffices if $4pD \leq 1$. Use the Asymmetric LLL to show the following slight strengthening of this: show that the condition "$ep(D+1) \leq 1$" is sufficient, where e is the base of the natural logarithm. **Hint:** Use the Asymmetric LLL, with all the values x_i being the *same*. Choose the "best" common value for the x_i that you can, that will satisfy (5).

3(a). Use the result of problem 2(b) to show the following. Suppose we have a hypergraph where each hyperedge has t elements, and where each vertex occurs in at most t hyperedges. Show that if $t \geq 9$, the hypergraph is 2-colorable. **Hint:** Do a random 2-coloring, and apply the result of problem 2(b).

(b) Strengthen the result of part (a) to show that it in fact holds for all $t \geq 7$. The idea is to combine the result of problem 2(b) with the definition (2), and is briefly as follows. Do a random coloring with colors red and blue, and for each hyperedge f, *separately* define the bad events $E_{f,B}$ and $E_{f,R}$, which stand for "f all blue" and "f all red" respectively. Now use (2) and the FKG inequality to suitably define the sets S_i, and proceed to use the result of problem 2(b) to show that with positive probability, all the bad events can be avoided.

4. Prove the second part of Janson's inequality (called the Generalized Janson inequality) from the first part (which we proved in class), as follows. Recall the basic setting: we have *independent* random variables $X_1, X_2, \ldots, X_n \in \{0, 1\}$, and also have subsets $S_1, S_2, \ldots, S_m \subseteq \{1, 2, \ldots, n\}$. Let I_i be the event "$\forall j \in S_i, X_j = 1$", and let D_i be the event $\overline{I_i}$. We want an upper-bound on $\Pr[D_1 \wedge D_2 \wedge \cdots \wedge D_m]$, and will proceed as follows, using the following more general notation than what we used in class. For a subset $A \subseteq \{1, 2, \ldots, m\}$, define

$$\mu(A) = \sum_{i \in A} \Pr[I_i]; \quad \Delta(A) = \sum_{i,j \in A:\ i \neq j,\ S_i \cap S_j \neq \emptyset} \Pr[I_i \wedge I_j].$$

We want to show that if $\mu \doteq \mu(\{1, 2, \ldots, m\})$ and $\Delta \doteq \Delta(\{1, 2, \ldots, m\})$ satisfy $\Delta \geq 2\mu$, then $\Pr[D_1 \wedge D_2 \wedge \cdots \wedge D_m] \leq e^{-\mu^2/(2\Delta)}$. Do a suitable random construction of a subset $A \subseteq \{1, 2, \ldots, m\}$, to show that there exists some A such that $-\mu(A) + \Delta(A)/2 \leq -\mu^2/(2\Delta)$. Next use the version of Janson's inequality proved in class, to prove that $\Pr[D_1 \wedge D_2 \wedge \cdots \wedge D_m] \leq e^{-\mu^2/(2\Delta)}$.