Fall 2002 Khuller & Postow

CMSC 351
Solutions to Homework 1

Solution 1.

$$\sum_{i=1}^{n}\sum_{j=i}^{3i+1} 1 = \sum_{i=1}^{n}(2i+2) = \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2 = 2\cdot\frac{n(n+1)}{2} + 2\cdot n = n(n+3)$$

Solution 2. Sort the numbers. Now assume that x_1 is the smallest, x_2 the next smallest and so on. Note that we should pair x_{2n} (largest number) with x_1 the smallest number. To see this formally, suppose that the OPT solution pairs x_{2n} with some number x_p and x_1 with x_j. By doing an exchange, and producing the pairing (x_{2n}, x_1) and (x_p, x_j) we get a solution with lower cost, since $x_{2n} + x_1 \leq x_{2n} + x_p$ and $x_p + x_j \leq x_{2n} + x_p$. We now continue to do the pairing in this way. The maximum pair that we produce is our cost.

Solution 3. We only give the solution for part (a). If you sort all the numbers using merge sort, then all the duplicate numbers appear consecutively. It is easy to compute the mode in a single scan of the numbers now. This algorithm takes time $O(n \log n)$.

Solution 4. It is easy to verify the claim for $n = 1$. The LHS is 1, and the RHS is $\frac{1.2.3}{6} = 1$. We now assume that the claim is true for $n = m$ and we will use this to prove that the claim is true for $n = m+1$. We need to prove that (setting $n = m+1$)

$$\sum_{i=1}^{m+1} i^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

Consider the LHS.

$$\sum_{i=1}^{m} i^2 + (m+1)^2.$$

This is $\frac{(m)(m+1)(2m+1)}{6} + (m+1)^2$. Simplifying, gives us $(m+1)\frac{2m^2+7m+6}{6}$. This is the same as the RHS. (Since $(m+2)(2m+3) = 2m^2 + 7m + 6$.)