Instructions: There are three questions. If you think you have some ideas that deserve partial credit, please itemize them concisely. Also, if you are unable to solve a problem but can do so by making some assumptions, clearly state these assumptions and then proceed. Good luck!

1. (4 points) Let $X$ be a non-negative random variable. We generate independent random variables $X_1, X_2, X_3, \ldots$, each of which has the same distribution as $X$. Let $I$ be the following random variable: $I$ is the smallest integer $i$ such that $X_i < 2 \cdot E[X]$. Given a positive integer $k$, show that $Pr[I > k] \leq 2^{-k}$.

2. (5 points) Choose a random permutation $\pi$ of $\{1, 2, \ldots, n\}$ (i.e., uniformly at random from the set of all $n!$ permutations). Then, for any nonempty subset $X$ of $\{1, 2, \ldots, n\}$, define a random variable $h(X) = \min_{x \in X} \pi(x)$.

(For instance, suppose $n = 4$, $\pi = (2, 4, 1, 3)$, and $X = \{1, 3\}$. Then, since $\pi(1) = 2$ and $\pi(3) = 1$, we have in this example that $h(X) = \min\{2, 1\} = 1$.)

Consider any two nonempty subsets $A$ and $B$ of $\{1, 2, \ldots, n\}$. Express the probability of the event “$h(A) = h(B)$” as a function of $|A \cap B|$ and $|A \cup B|$. (As usual, $|X|$ denotes the cardinality of a finite set $X$. Note that the only underlying random process in this problem is the random choice of $\pi$.)

3. (6 points) We are given some constant $\epsilon$, where $0 < \epsilon \leq 1/2$. A peer $p$ in Chord wants to estimate $n$, the total number of peers, to within a high accuracy: it wants to compute a number $n'$ such that $(1 - \epsilon)n \leq n' \leq (1 + \epsilon)n$ with high probability. Show how to build on the protocol of King and Saia, to achieve this using only $O(\log n)$ communication; the constant hidden in the $O()$ notation can be a function of $\epsilon$, and you can assume that $n$ is large enough. (Hint: King and Saia show how to estimate $n$ to within some fixed – not arbitrary – constant factor. Start with such an estimate, and see how you can use ideas from the King-Saia protocol.)

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