CMSC 858S: Algorithms in Networking, Fall 2004

Mid-Term Exam: 11 AM – 12:15 PM, October 21 2004

Instructions: There are three questions. If you think you have some ideas that deserve partial credit, please *itemize them concisely.* Also, if you are unable to solve a problem but can do so by making some assumptions, clearly state these assumptions and then proceed. Good luck!

1. **(4 points)** Let X be a *non-negative* random variable. We generate *independent* random variables $X_1, X_2, X_3, \ldots$, each of which has the same distribution as X. Let I be the following random variable: I is the smallest integer i such that $X_i < 2 \cdot \mathbf{E}[X]$. Given a positive integer k, show that $\Pr[I > k] \leq 2^{-k}$.

2. **(5 points)** Choose a random permutation π of $\{1, 2, \ldots, n\}$ (i.e., uniformly at random from the set of all $n!$ permutations). Then, for any nonempty subset X of $\{1, 2, \ldots, n\}$, define a random variable

$$h(X) = \min_{x \in X} \pi(x).$$

(For instance, suppose $n = 4$, $\pi = (2, 4, 1, 3)$, and $X = \{1, 3\}$. Then, since $\pi(1) = 2$ and $\pi(3) = 1$, we have in this example that $h(X) = \min\{2, 1\} = 1$.)

Consider any two nonempty subsets A and B of $\{1, 2, \ldots, n\}$. Express the probability of the event "$h(A) = h(B)$" as a function of $|A \cap B|$ and $|A \cup B|$. (As usual, $|X|$ denotes the cardinality of a finite set X. Note that the only underlying random process in this problem is the random choice of π.)

3. **(6 points)** We are given some constant ϵ, where $0 < \epsilon \leq 1/2$. A peer p in Chord wants to estimate n, the total number of peers, to within a high accuracy: it wants to compute a number n' such that $(1 - \epsilon)n \leq n' \leq (1 + \epsilon)n$ with high probability. Show how to build on the protocol of King and Saia, to achieve this using only $O(\log n)$ communication; the constant hidden in the $O()$ notation can be a function of ϵ, and you can assume that n is large enough. (**Hint:** King and Saia show how to estimate n to within some fixed – not arbitrary – constant factor. Start with such an estimate, and see how you can use ideas from the King-Saia protocol.)

– End of Exam –