CMSC 858S: Algorithms in Networking

Fall 2004

Homework Assignment #1 **(will be graded)**

Due date: Beginning of class on September 30, 2004

Note: As with all homework assignments for this class, this needs to be done along with your group-mates. Please submit one final, polished set of answers for your group, and do not include unnecessary material. (For instance, don't include your preliminary work/calculations/intuition that led to the final answer.) Partial credit will be given where appropriate: if you think you have some ideas that deserve partial credit, please *itemize them concisely*, so that the grading process can be accurate.

The problem marked (*) may be more difficult than the others. Also, the Chernoff-bound approach of bounding: (i) $\Pr[X \geq a]$ by $\Pr[e^{tX} \geq e^{ta}]$, and (ii) $\Pr[X \leq a]$ by $\Pr[e^{-tX} \geq e^{-ta}]$ (and then applying Markov, simplifying, and choosing the optimal positive t), is quite general: the case where X is a sum of bounded and independent random variables seen in class, is just a special case. Problems 2 and 3 deal with such generalizations.

1. We have a random variable X whose distribution is some D; our aim is to estimate $\mathbf{E}[X]$ using "not too many" samples from D. Suppose we can draw independent random samples $X_1, X_2, \ldots$ from D; we want to draw some t such samples, and output the estimate $Y = (X_1 + X_2 + \cdots + X_t)/t$ for $\mathbf{E}[X]$. Our **main goal** is that with probability at least $1 - \delta$, the absolute value of the difference between our estimate and $\mathbf{E}[X]$, should be at most $\epsilon \cdot \mathbf{E}[X]$.

We now explore how to solve this problem. Suppose the value $\ell = \left\lceil \sqrt{\mathrm{Var}[X]}/\mathbf{E}[X] \right\rceil$ is given, but that we do not know anything more about D.

(P1) Show that $O(\ell^2/(\epsilon^2\delta))$ samples suffice. (We will see how to improve this in (P3).)

(P2) Suppose we only need a "weak estimate" Z such that: with probability at least 0.6, the absolute value of the difference between Z and $\mathbf{E}[X]$, is at most $\epsilon \cdot \mathbf{E}[X]$. Show that $O(\ell^2/\epsilon^2)$ samples suffice to compute such a weak estimate.

(P3) Show that by judiciously combining $O(\log(1/\delta))$ independently-computed weak estimates, we can solve our main problem: i.e., we can get an estimate such that with probability at least $1 - \delta$, the absolute value of the difference between our estimate and $\mathbf{E}[X]$, is at most $\epsilon \cdot \mathbf{E}[X]$. (Thus, $O(\ell^2 \log(1/\delta)/\epsilon^2)$ samples suffice. A hint: the mean may not be the correct choice for the "judicious combination"!)

2. Suppose X is a Poisson random variable with mean λ; i.e., X takes values in the non-negative integers, with $\Pr[X = i] = e^{-\lambda} \cdot \lambda^i / i!$. Given $\delta > 0$, use a Chernoff-type approach to bound $\Pr[X \geq \lambda(1 + \delta)]$. Does your result resemble any bound seen in class?

3 (*). We are given a black-box that outputs numbers independently and uniformly at random from the set $\{1, 2, \ldots, 2n\}$. We use this to construct a random permutation π of $\{1, 2, \ldots, n\}$ as follows. We will construct a one-to-one function $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, 2n\}$, and then output the numbers in $\{1, 2, \ldots, n\}$ in the order of their $f(\cdot)$ values. To construct f, we proceed as follows, for $j = 1, 2, \ldots, n$: choose $f(j)$ by repeatedly obtaining numbers from the black-box, and setting $f(j)$ to be the first number found such that $f(j) \neq f(i)$ for all $i < j$. (Note that we ensure that f is one-to-one.)

Let X denote the number of calls to the black-box.

- Prove that we output a permutation that is chosen uniformly at random from the set of all $n!$ permutations.
- Find the value of $\mathbf{E}[X]$.
- Show that there is a constant $C > 0$ such that for all n large enough, $\Pr[X \geq 2 \cdot \mathbf{E}[X]] \leq e^{-Cn}$.