

# CMSC 858S: Algorithms in Networking

Fall 2004

Ungraded Homework Assignment #1, handed out Sep. 7, 2004

**Note:** We will have *ungraded* homework assignments such as this one, as well as ones that will be graded. I will post the solutions for all the assignments some time after they are handed out. You will get the most out of this course if you do your best to solve all the homework problems (whether they are graded or not) in collaboration with your group-members. In addition, I hope the collaboration will be a rewarding experience in co-operative work for you; please let me know if you have found simpler or more elegant solutions than the ones handed out. Problems 4, 5, and 6 may be harder than the others. The suggested deadline by which to finish this assignment is September 16th; since this assignment is ungraded, you don't need to turn it in – just compare your solutions with the solutions I give.

1. A coin has probability  $p$  of coming up Heads. We keep tossing it until we first see Heads. What is the expected number of coin-tosses we make?

2. Consider the following random process, where we keep choosing numbers from the set  $S = \{1, 2, \dots, n\}$  in steps; in each step, we choose a random element of  $S$ . What is the expected number of steps we need until all elements of  $S$  have been generated?

3. Suppose we use some probability  $p$  of being awake in every step, in Luby's algorithm. Do the same analysis as done in class; as far as this analysis is concerned, is  $p = 1/2$  the best choice?

4. We derived Chebyshev's inequality by using Markov's inequality with a certain quadratic function of the random variable  $X$  whose tail we wish to bound. Now suppose  $\mathbf{E}[X] = \mu$ ,  $X$  has variance  $\sigma^2$ , some  $a > 0$  is given, and we wish to bound only the upper-tail  $\Pr[X \geq \mu + a]$ . Construct another "optimal" quadratic function of  $X$  and use Markov's inequality, to show that  $\Pr[X \geq \mu + a] \leq \sigma^2/(\sigma^2 + a^2)$ . Furthermore:

- Derive the same upper-bound for the lower tail,  $\Pr[X \leq \mu - a]$ .
- These upper- and lower-tail bounds constitute the *Chebyshev-Cantelli inequality*. When are they better than Chebyshev's inequality, which gives  $\Pr[|X - \mu| \geq a] \leq \sigma^2/a^2$ ?

5. Suppose we implement the probabilistic proof of Turán's theorem as discussed in class, for a given graph  $G = (V, E)$ : each vertex  $v$  chooses a random real  $x_v \in [0, 1]$ , and goes into the set  $I$  iff  $x_v < x_w$  for all neighbors  $w$  of  $v$ . Let  $Y$  denote the cardinality of  $I$ . Suppose we only consider graphs  $G$  whose maximum degree is bounded by some **constant**  $d$ . Then, for any constant  $\epsilon > 0$ , use the Second Moment Method to show that

$$\Pr[|Y - \mathbf{E}[Y]| \geq \epsilon \cdot \mathbf{E}[Y]]$$

approaches zero as  $|V|$  becomes large.

6. Suppose we run the process of Problem 2 for  $t = \lceil n \ln n \rceil$  steps. Let  $X$  denote the total number of distinct elements of  $S$  chosen in these  $t$  steps. Use one of the tail inequalities that has been covered in class or in this homework, to show the following: there is a constant  $c < 1$  such that as  $n$  gets large,  $\Pr[X = n] \leq c$ . (**Hints:** (a) Let  $X_i$  be the indicator random variable for  $i \in S$  being chosen in the  $t$  steps. Intuitively, if  $i \neq j$ , then  $X_i$  and  $X_j$  are "negatively correlated". Try to formalize and use this. (b) Show that  $\lim_{n \rightarrow \infty} \frac{(1-1/n)^t}{1/n} = 1$ . This may simplify some of your computations.)