Heaps & Priority Queues

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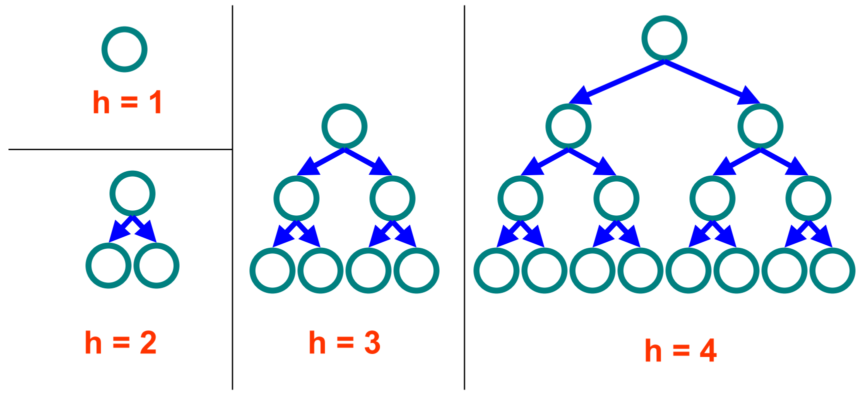
Overview

- Binary trees
  - Perfect
  - Complete
- Heaps
- Priority queues

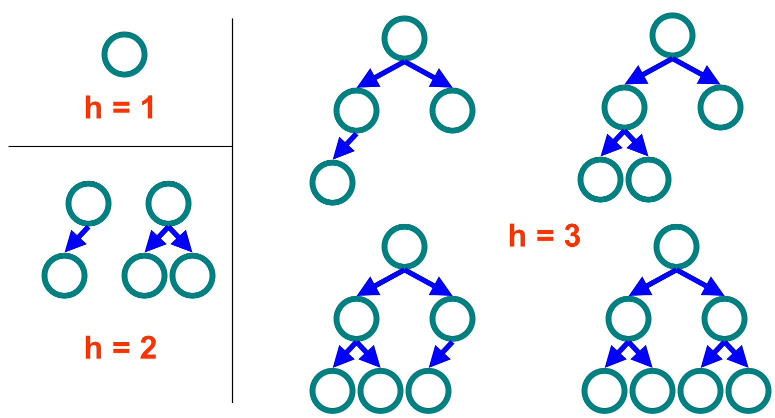
Perfect Binary Tree

- **For binary tree with height h**
  - **All nodes at levels h–1 or less have 2 children (full)**



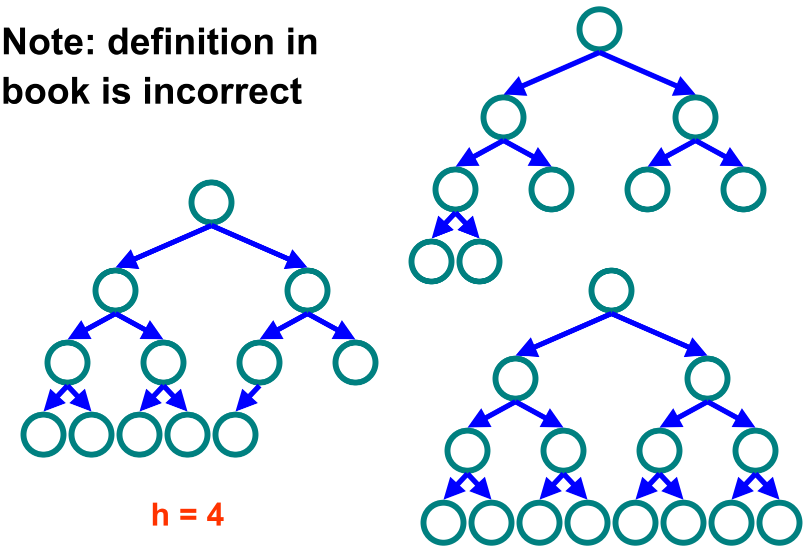
Complete Binary Trees

- **For binary tree with height h**
  - **All nodes at levels h–2 or less have 2 children (full)**
  - **All leaves on level h are as far left as possible**



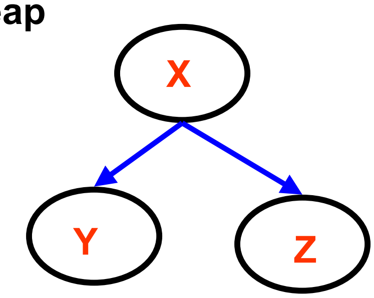
Complete Binary Trees

- Note: definition in book is incorrect

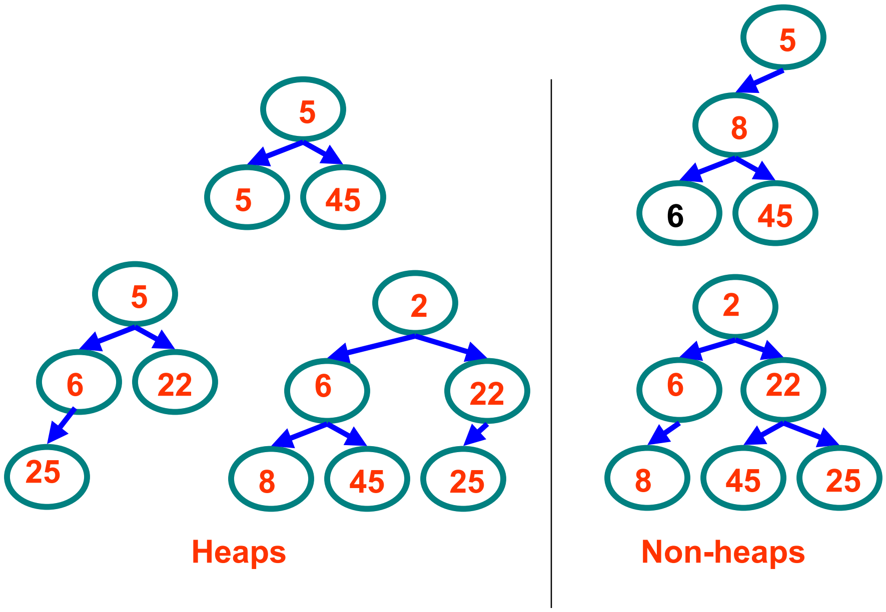


Heaps

- Two key properties
  - Complete binary tree
  - Value at node
    - Smaller than or equal to values in subtrees
- Example heap
  - $X \leq Y$
  - $X \leq Z$



Heap & Non-heap Examples



Heap Properties

- **Key operations**
  - **Insert (X)**
  - **getSmallest ( )**
- **Key applications**
  - **Heapsort**
  - **Priority queue**

Heap Operations – Insert(X)

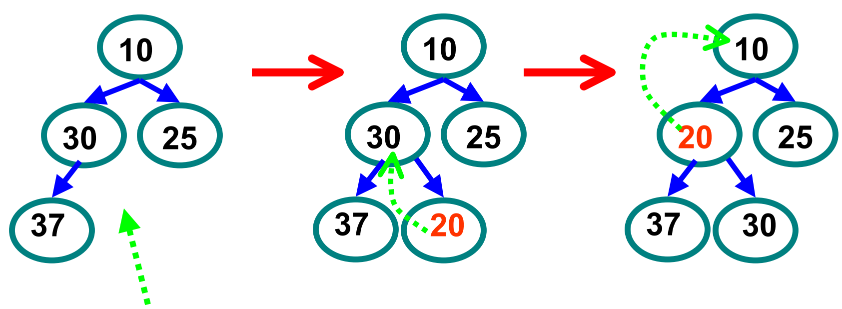
- Algorithm
  1. Add X to end of tree
  2. While (X < parent)

     Swap X with parent // X bubbles up tree

- Complexity
  - # of swaps proportional to height of tree
  - O(log(n))

Heap Insert Example

- Insert (20)



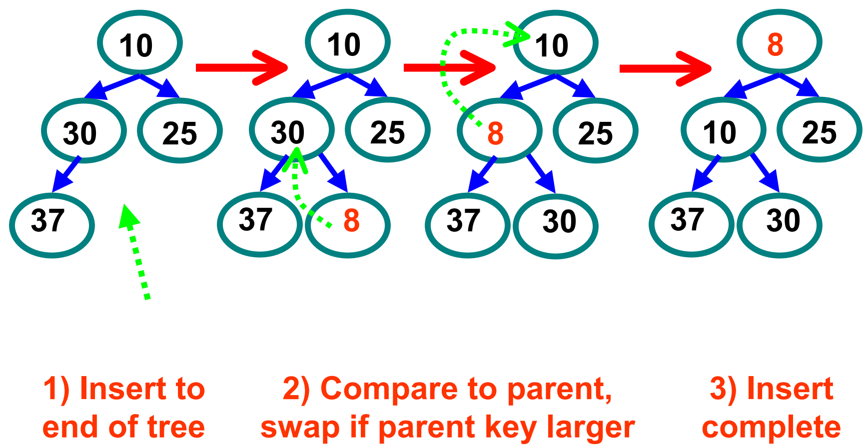
1) Insert to end of tree

2) Compare to parent, swap if parent key larger

3) Insert complete

Heap Insert Example

- Insert (8)



Heap Operation – getSmallest()

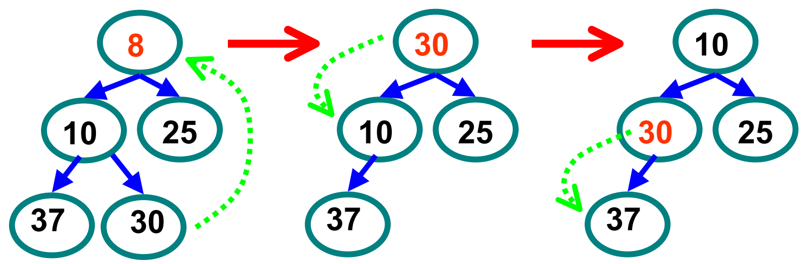
- Algorithm
  1. Get smallest node at root
  2. Replace root with X at end of tree
  3. While (X > child)

     Swap X with smallest child // X drops down tree
  4. Return smallest node

- Complexity
  - # swaps proportional to height of tree
  - O(log(n))

Heap GetSmallest Example

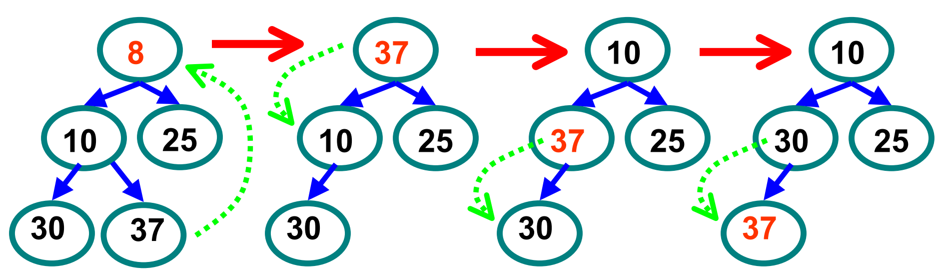
- **getSmallest ()**



1) Replace root with end of tree **2) Compare node to children, if larger swap with smallest child** **3) Repeat swap if needed**

Heap GetSmallest Example

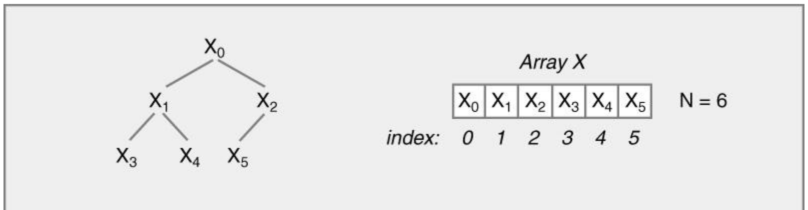
- **getSmallest ()**



1) Replace root with end of tree **2) Compare node to children, if larger swap with smallest child** **3) Repeat swap if needed**

Heap Implementation

- Can implement heap as array
  - Store nodes in array elements
  - Assign location (index) for elements using formula



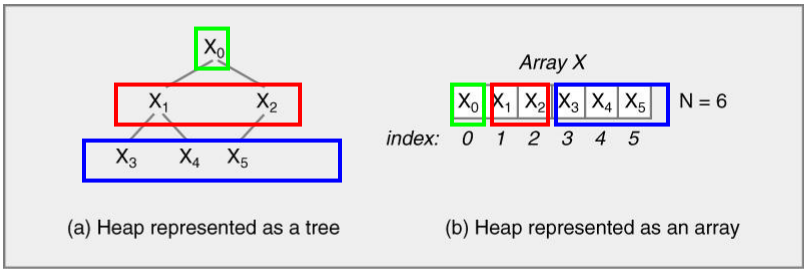
(a) Heap represented as a tree (b) Heap represented as an array

Heap Implementation

- Observations
  - Compact representation
  - Edges are implicit (no storage required)
  - Works well for complete trees (no wasted space)

Heap Implementation

- **Calculating node locations**
  - **Array index i starts at 0**
  - **Parent(i) = ⌊ (i – 1) / 2 ⌋**
  - **LeftChild(i) = 2 × i +1**
  - **RightChild(i) = 2 × i +2**

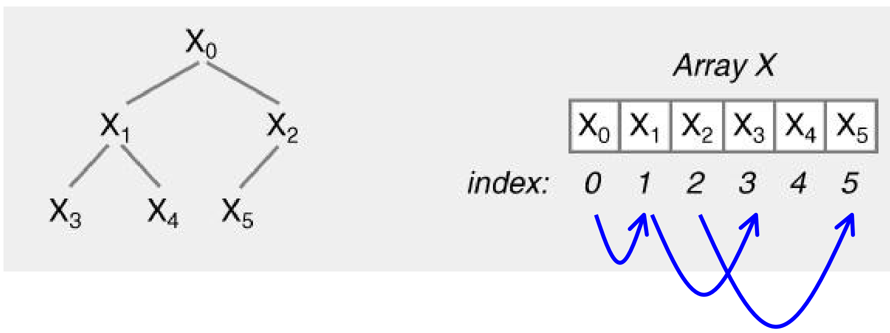


Heap Implementation

- **Example**
  - **Parent(1) = ⌊ (1 – 1) / 2 ⌋ = ⌊ 0 / 2 ⌋ = 0**
  - **Parent(2) = ⌊ (2 – 1) / 2 ⌋ = ⌊ 1 / 2 ⌋ = 0**
  - **Parent(3) = ⌊ (3 – 1) / 2 ⌋ = ⌊ 2 / 2 ⌋ = 1**
  - **Parent(4) = ⌊ (4 – 1) / 2 ⌋ = ⌊ 3 / 2 ⌋ = 1**
  - **Parent(5) = ⌊ (5 – 1) / 2 ⌋ = ⌊ 4 / 2 ⌋ = 2**

Heap Implementation

- **Example**
  - **LeftChild(0) = 2 × 0 +1 = 1**
  - **LeftChild(1) = 2 × 1 +1 = 3**
  - **LeftChild(2) = 2 × 2 +1 = 5**



Heap Implementation

- **Example**
  - **RightChild(0) = 2 × 0 +2 = 2**
  - **RightChild(1) = 2 × 1 +2 = 4**

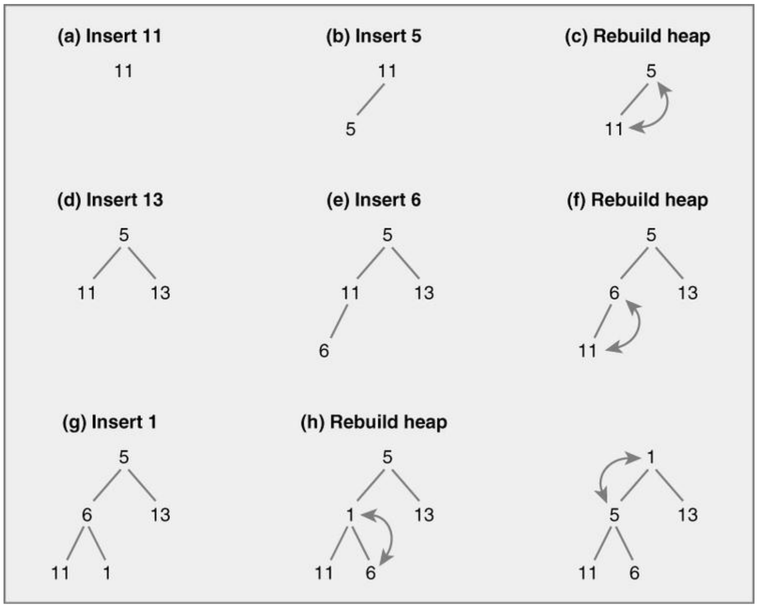
Heap Application – Heapsort

- Use heaps to sort values
  - Heap keeps track of smallest element in heap
- Algorithm
  1. Create heap
  2. Insert values in heap
  3. Remove values from heap (in ascending order)
- Complexity
  - O(nlog(n))

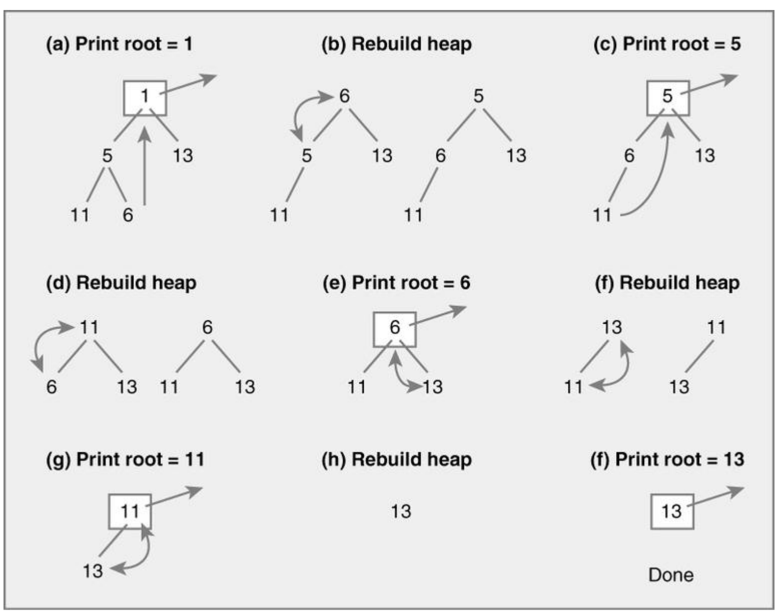
Heapsort Example

- Input
  - 11, 5, 13, 6, 1
- View heap during insert, removal
  - As tree
  - As array

Heapsort – Insert Values



Heapsort – Remove Values



Heapsort – Insert in to Array 1

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Insert 11	11				

Heapsort – Insert in to Array 2

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Insert 5	11	5			
Swap	5	11			

Heapsort – Insert in to Array 3

- **Input**
  - **11, 5, 13, 6, 1**

Index =	0	1	2	3	4
Insert 13	5	11	13		

Heapsort – Insert in to Array 4

- **Input**
  - **11, 5, 13, 6, 1**

Index =	0	1	2	3	4
Insert 6	5	11	13	6	
Swap	5	6	13	11	

…

Heapsort – Remove from Array 1

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Remove root	1	5	13	11	6
Replace	6	5	13	11	
Swap w/ child	5	6	13	11	

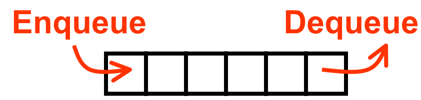
Heapsort – Remove from Array 2

- Input
  - 11, 5, 13, 6, 1

Index =	0	1	2	3	4
Remove root	5	6	13	11	
Replace	11	6	13		
Swap w/ child	6	11	13		

Heap Application – Priority Queue

- **Queue**
  - **Linear data structure**
  - **First-in First-out (FIFO)**
  - **Implement as array / linked list**



- **Priority queue**
  - **Elements are assigned priority value**
  - **Higher priority elements are taken out first**
  - **Equal priority elements are taken out in FIFO order**
  - **Implement as heap**

Priority Queue

- **Properties**
  - **Lower value = higher priority**
  - **Heap keeps highest priority items in front**
- **Complexity**
  - **Enqueue (insert) = O(log(n))**
  - **Dequeue (remove) = O(log(n))**
  - **For any heap**

Heap vs. Binary Search Tree

- Binary search tree
  - Keeps values in sorted order
  - Find any value
    - O(log(n)) for balanced tree
    - O(n) for degenerate tree (worst case)
- Heap
  - Keeps smaller values in front
  - Find minimum value
    - O(log(n)) for any heap
  - Can also organize heap to find maximum value
    - Keep largest value in front