Biconnected Components

Let $G = (V, E)$ be a connected, undirected graph. A vertex a is called an *articulation vertex/cut vertex* if the deletion of a from G (together with its incident edges) makes the graph disconnected (creates two or more connected components). Equivalently, there must exist two vertices v and w (distinct from a) such that every path from v to w goes through a.

A graph G is called biconnected if for every distinct triple of vertices v, w, a there exists a path between v and w not containing a. In other words, a graph is biconnected if and only if it has no articulation vertices.

We can now define a natural relation on the set of edges of G, by saying that two edges e_1 and e_2 are related if $e_1 = e_2$ or there is a cycle containing both e_1 and e_2. It is not hard to show that this is an equivalence relation that partitions the edges of G into equivalence classes $E_1, E_2, \ldots, E_k$ such that two distinct edges are in the same class if and only if they lie on a common cycle. Let V_i be the set of vertices "touched" by edges in E_i. Then the graph $G_i = (V_i, E_i)$ is called a *biconnected component* of G.
Note: This is the equivalent to the definition (given in class) that a biconnected component is a maximal subset of edges such that its induced subgraph is biconnected.

Theorem 1 *For $1 \leq i \leq k$, let $G_i = (V_i, E_i)$ be the biconnected component of a connected undirected graph $G = (V, E)$. Then*

1. *G_i is biconnected.*
2. *For all $i \neq j$, $V_i \cap V_j$ contains atmost one vertex.*
3. *a is an articulation vertex if and only if $a \in V_i \cap V_j$ for some $i \neq j$.*

Proof:

1. Suppose that a is an articulation vertex in V_i. Let v and w be two vertices such that all paths from v to w pass through a. Surely (v, w) is not an edge in E_i. Thus there are distinct edges (v, v') and (w, w') in E_i and there is a cycle through these edges. (This is the case since v, w are vertices in the graph induced by E_i, which was defined as a collection of edges that had a cycle through any two of them.) Thus there are two disjoint paths from v to w, and only one of them can contain a.

2. Suppose that there are two distinct vertices v and w in $V_i \cap V_j$. Then there exists a cycle C_1 in G_i that contains v and w, and a cycle C_2 in G_j that also contains v and w. Since E_i and E_j are disjoint, the sets of edges in C_1 and C_2 are disjoint. However we can now construct a cycle containing v and w that uses edges from C_1 and C_2 implying that E_i and E_j are not equivalence classes. (Since an edge in E_i is equivalent to an edge in E_j.)

3. Suppose that a is an articulation vertex of G. Then there are two vertices v, w such that all paths from v to w pass through a. Let (x, a) and (y, a) be the two edges on a path between v and w incident on a. If there is a cycle through these two edges then there is a path from v to w that does not contain a. Thus the two edges are in different biconnected components, and a is in the intersection of their vertex sets.
   For the converse, if $a \in V_i \cap V_j$, then there are edges (x, a) and (y, a) in E_i and E_j respectively. Since these two edges do not occur on any single cycle, it follows that every path from x to y contains a. Thus a is an articulation vertex.