Toothpick Game

There are two players, A (Amanda) and B (Billy). They move alternately. In each move, A can pick either 1 or 4 toothpicks from the table. In each move, B can pick either 1, 3 or 5 toothpicks from the table. The winner of the game is the person who gets the last toothpick. (If there are 2 toothpicks, and for example its A's move – then A must pick 1 toothpick, since it is not possible to pick 4 toothpicks.)

How do we find a winning strategy for A (if one exists)? We are interested in doing this for A since she is the smarter of the two, and knows dynamic programming!

We do this by defining two functions f and g (value of $f(n)$ and $g(n)$ is 0 or 1).

$f(n) = 1$ iff there is a winning strategy for A given that it is A's move and there are n toothpicks

$g(n) = 1$ iff there is a winning strategy for A given that it is B's move and there are n toothpicks

Assuming $n > 5$ (we can easily compute the function for small values of n) we can define f and g recursively as follows.

$$f(n) = \max(g(n-1), g(n-4)).$$

This is because A has a choice of picking either 1 or 4 toothpicks, clearly A will make the move that enables it to continue winning after its move (the function g gives us this information).

$$g(n) = \min(f(n-1), f(n-3), f(n-5)).$$

We make the assumption that B will also be playing intelligently. If B makes an error (for example B picks 1 toothpick when $f(n-1) = 1$ and $f(n-3) = 0$, then from that point on A will win; clearly, picking 3 toothpicks was the right choice for B).

We can use the above equation to compute the f and g values (fill the rest of the table yourself).

	1	2	3	4	5	6	7	8	9	10	11
f	1	0	1	1	1	1	1				
g	0	1	0	1	0	1	0				
A move	1		1	4	1	4	1				
B move	1		3		5		5				

n ⟶

Figure 1: Example for small values of n.

Observe that if $f(n) = 1$ then the corresponding entry in the 3rd row tells A what move to make. If $f(n) = 0$ then it does not matter what move A makes (since neither move can guarantee a win). If $g(n) = 0$ then A cannot guarantee a win. As a result, the corresponding entry in the 4th row tells B what move to make. If $g(n) = 1$ then it does not matter what move B makes (neither move can guarantee B a win).