

### A Short Note on Colliding Spheres

The Law of Momentum Conservation is one of the most important physical laws in determining the physical motion of objects. As you may recall from basic physics, momentum  $\mathbf{p}$  is defined as the product of the mass  $m$  of an object and its velocity  $\mathbf{v}$ . Before and after collisions, the sum of momentums is preserved. If the collision is totally elastic (think of ping-pong balls), then the kinetic energy is conserved as well.

Let us consider a simple case in which objects are spheres and have the same masses.<sup>1</sup> Let us do this in the 1-dimensional case first, where the two objects are moving along a common line. Suppose that there are two object  $O_1$  and  $O_2$  with identical masses  $m$  and their velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively. And after collision, let  $\mathbf{v}'_1$  and  $\mathbf{v}'_2$  denote their new respective velocities. By the law of conservation momentum we have

$$m\mathbf{v}_1 + m\mathbf{v}_2 = m\mathbf{v}'_1 + m\mathbf{v}'_2.$$

By the law of conservation kinetic energy we have

$$\frac{1}{2}m\mathbf{v}_1^2 + \frac{1}{2}m\mathbf{v}_2^2 = \frac{1}{2}m(\mathbf{v}'_1)^2 + \frac{1}{2}m(\mathbf{v}'_2)^2.$$

Solving these equations, we find that the two velocities have swapped. That is,

$$\mathbf{v}'_1 = \mathbf{v}_2 \quad \text{and} \quad \mathbf{v}'_2 = \mathbf{v}_1.$$

Now, let us apply this to the 3-d collision of two equal spheres. Because these are spheres, the impact point and the object centers are collinear along some line, call it  $\ell$ . We can decompose the motion into two vector components, one parallel to  $\ell$  and one orthogonal to  $\ell$ . The response in the parallel components of velocity follows from the 1-dimensional case. Assuming a frictionless collision, the orthogonal velocity components are unaffected by the collision

To make this more formal, let  $\mathbf{u}$  denote an unit length vector from the center of  $O_1$  to the center of  $O_2$ . That is,

$$\mathbf{u} = \frac{O_2 - O_1}{\|O_2 - O_1\|}.$$

Recalling that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  denote the respective velocities of  $O_1$  and  $O_2$ , we can decompose them into their components that are parallel to and orthogonal to  $\mathbf{u}$ . By basic linear algebra we have:

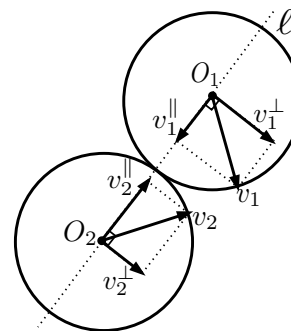
$$\mathbf{v}_1^{\parallel} = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{u} \quad \text{and} \quad \mathbf{v}_1^{\perp} = \mathbf{v}_1 - \mathbf{v}_1^{\parallel}.$$

(Recall that  $(\mathbf{u} \cdot \mathbf{v}) = u_x v_x + u_y v_y + u_z v_z$ .) We can do the same for  $\mathbf{v}_2$ , yielding:

$$\mathbf{v}_2^{\parallel} = (\mathbf{u} \cdot \mathbf{v}_2)\mathbf{u} \quad \text{and} \quad \mathbf{v}_2^{\perp} = \mathbf{v}_2 - \mathbf{v}_2^{\parallel}.$$

From our earlier observation on the 1-dimensional case we find that the new velocities after collision are:

$$\mathbf{v}'_1 = \mathbf{v}_2^{\parallel} + \mathbf{v}_1^{\perp} \quad \text{and} \quad \mathbf{v}'_2 = \mathbf{v}_1^{\parallel} + \mathbf{v}_2^{\perp}.$$



<sup>1</sup>The general case of unequal masses is discussed in the web page <http://en.wikipedia.org/wiki/Momentum>.