CMSC 132: Object-Oriented Programming II

Algorithmic Complexity I

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Algorithm Efficiency

- Efficiency
  - Amount of resources used by algorithm
    - Time, space
- Measuring efficiency
  - Benchmarking
  - Asymptotic analysis

Benchmarking

- Approach
  - Pick some desired inputs
  - Actually run implementation of algorithm
  - Measure time & space needed
- Industry benchmarks
  - SPEC – CPU performance
  - MySQL – Database applications
  - WinStone – Windows PC applications
  - MediaBench – Multimedia applications
  - Linpack – Numerical scientific applications

Benchmarking

- Advantages
  - Precise information for given configuration
    - Implementation, hardware, inputs
- Disadvantages
  - Affected by configuration
    - Data sets (usually too small)
    - Hardware
    - Software
  - Affected by special cases (biased inputs)
  - Does not measure intrinsic efficiency

Asymptotic Analysis

- Approach
  - Mathematically analyze efficiency
  - Calculate time as function of input size n
    - $T \approx O[\ f(n)\ ]$
    - T is on the order of f(n)
    - "Big O" notation
- Advantages
  - Measures intrinsic efficiency
  - Dominates efficiency for large input sizes

Search Example

- Number guessing game
  - Pick a number between 1…n
  - Guess a number
  - Answer "correct", "too high", "too low"
  - Repeat guesses until correct number guessed

Linear Search Algorithm

- Algorithm
  1. Guess number = 1
  2. If incorrect, increment guess by 1
  3. Repeat until correct
- Example
  - Given number between 1…100
  - Pick 20
  - Guess sequence = 1, 2, 3, 4 … 20
  - Required 20 guesses

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Linear Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = 1, requires 1 guess
  - If number = n, requires n guesses
  - On average, needs n/2 guesses
  - Time = O(n) = Linear time

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Binary Search Algorithm

- Algorithm
  - Set Δ to n/4
  - Guess number = n/2
  - If too large, guess number – Δ
  - If too small, guess number + Δ
  - Reduce Δ by ½
  - Repeat until correct

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Binary Search Algorithm

- Example
  - Given number between 1…100
  - Pick 20
  - Guesses =
    - 50, Δ = 25, Answer = too large, subtract Δ
    - 25, Δ = 12 , Answer = too large, subtract Δ
    - 13, Δ = 6, Answer = too small, add Δ
    - 19, Δ = 3, Answer = too small, add Δ
    - 22, Δ = 1, Answer = too large, subtract Δ
    - 21, Δ = 1, Answer = too large, subtract Δ
    - 20
  - Required 7 guesses

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Binary Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = n/2, requires 1 guess
  - If number = 1, requires $\log_2(n)$ guesses
  - If number = n, requires $\log_2(n)$ guesses
  - On average, needs $\log_2(n)$ guesses
  - Time = $O(\log_2(n))$ = Log time

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Search Comparison

- For number between 1…100
  - Simple algorithm = 50 steps
  - Binary search algorithm = $\log_2(n)$ = 7 steps
- For number between 1…100,000
  - Simple algorithm = 50,000 steps
  - Binary search algorithm = $\log_2(n)$ (about 17 steps)
- Binary search is much more efficient!

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Asymptotic Complexity

- Comparing two linear functions

Size	Running Time	
	n/2	4n+3
64	32	259
128	64	515
256	128	1027
512	256	2051

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Asymptotic Complexity

- Comparing two functions
  - n/2 and 4n+3 behave similarly
  - Run time roughly doubles as input size doubles
  - Run time increases linearly with input size
- For large values of n
  - Time(2n) / Time(n) approaches exactly 2
- Both are O(n) programs

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Asymptotic Complexity

- Comparing two log functions

Size	Running Time	
	$\log_2(n)$	$5 * \log_2(n) + 3$
64	6	33
128	7	38
256	8	43
512	9	48

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Asymptotic Complexity

- Comparing two functions
  - $\log_2(n)$ and $5 * \log_2(n) + 3$ behave similarly
  - Run time roughly increases by constant as input size doubles
  - Run time increases logarithmically with input size
- For large values of n
  - Time(2n) – Time(n) approaches constant
  - Base of logarithm does not matter
    - Simply a multiplicative factor

      $\log_a N = (\log_b N) / (\log_b a)$
- Both are O(log(n)) programs

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Asymptotic Complexity

- Comparing two quadratic functions

Size	Running Time	
	n^2	$2n^2 + 8$
2	4	16
4	16	40
8	64	132
16	256	520

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Asymptotic Complexity

- Comparing two functions
  - n^2 and $2n^2 + 8$ behave similarly
  - Run time roughly increases by 4 as input size doubles
  - Run time increases quadratically with input size
- For large values of n
  - Time(2n) / Time(n) approaches 4
- Both are O(n^2) programs

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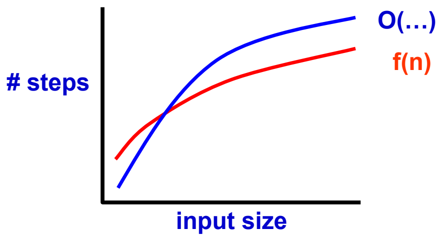
Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
    - For sufficiently large input size
  - Intrinsic efficiency of algorithm for large inputs



Formal Definition of Big-O

- Function f(n) is O(g(n)) if
  - For some positive constants M, N_0
  - $M \times g(n) \geq f(n)$, for all $n \geq N_0$
- Intuitively
  - For some coefficient M & all data sizes $\geq N_0$
    - $M \times g(n)$ is always greater than f(n)

Big-O Examples

- $5n + 1000 \Rightarrow O(n)$
  - Select M = 6, N_0 = 1000
  - For $n \geq 1000$
    - $6n \geq 5n+1000$ is always true
  - Example $\Rightarrow$ for n = 1000
    - $6000 \geq 5000 + 1000$

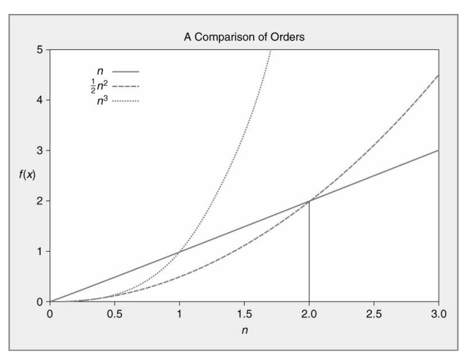
Big-O Examples

- $2n^2 + 10n + 1000 \Rightarrow O(n^2)$
  - Select M = 4, N_0 = 100
  - For $n \geq 100$
    - $4n^2 \geq 2n^2 + 10n + 1000$ is always true
  - Example $\Rightarrow$ for n = 100
    - $40000 \geq 20000 + 1000 + 1000$

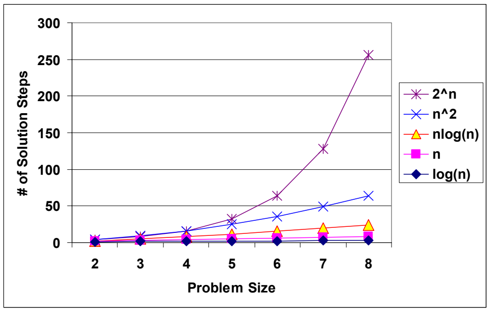
Observations

- Big O categories
  - O(log(n))
  - O(n)
  - $O(n^2)$
- For large values of n
  - Any O(log(n)) algorithm is faster than O(n)
  - Any O(n) algorithm is faster than $O(n^2)$
- Asymptotic complexity is fundamental measure of efficiency

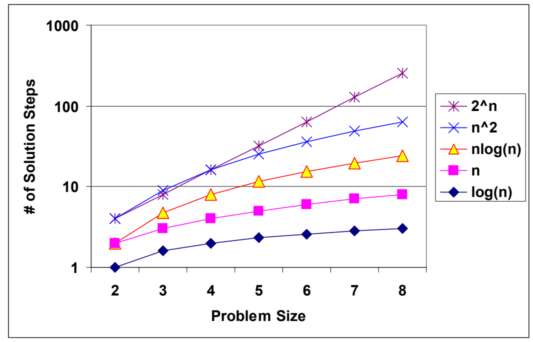
Comparison of Complexity



Complexity Category Example



Complexity Category Example

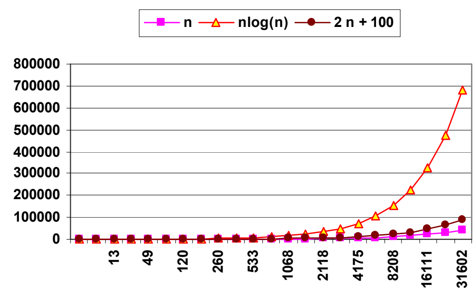


Calculating Asymptotic Complexity

- As n increases
  - Highest complexity term dominates
  - Can ignore lower complexity terms
- Examples
  - 2 n + 100 ⇒ O(n)
  - n log(n) + 10 n ⇒ O(nlog(n))
  - ½ n^2 + 100 n ⇒ O(n^2)
  - n^3 + 100 n^2 ⇒ O(n^3)
  - 1/100 2^n + 100 n^4 ⇒ O(2^n)

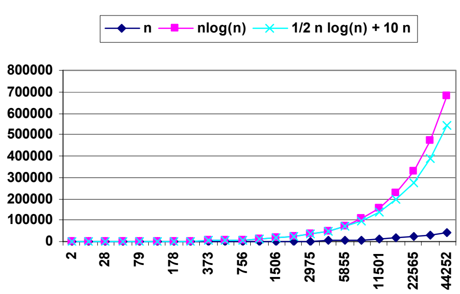
Complexity Examples

- 2n + 100 ⇒ O(n)



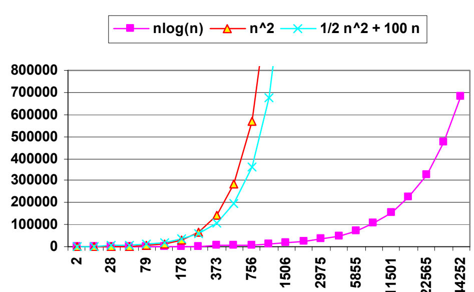
Complexity Examples

- ½ n log(n) + 10 n ⇒ O(nlog(n))



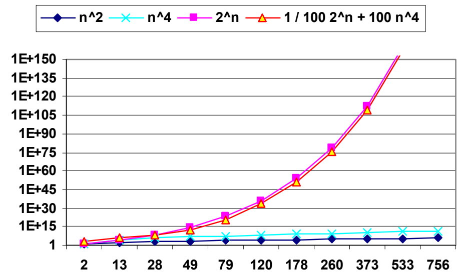
Complexity Examples

- ½ n^2 + 100 n ⇒ O(n^2)



Complexity Examples

- $1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$



Types of Case Analysis

- Can analyze different types (cases) of algorithm behavior
- Types of analysis
  - Best case
  - Worst case
  - Average case
  - Amortized

Types of Case Analysis

- Best case
  - Smallest number of steps required
  - Not very useful
  - Example ⇒ Find item in first place checked

Types of Case Analysis

- Worst case
  - Largest number of steps required
  - Useful for upper bound on worst performance
    - Real-time applications (e.g., multimedia)
    - Quality of service guarantee
  - Example ⇒ Find item in last place checked

Quicksort Example

- Quicksort
  - One of the fastest comparison sorts
  - Frequently used in practice
- Quicksort algorithm
  - Pick pivot value from list
  - Partition list into values smaller & bigger than pivot
  - Recursively sort both lists

Quicksort Example

- Quicksort properties
  - Average case = $O(n\log(n))$
  - Worst case = $O(n^2)$
    - Pivot ≈ smallest / largest value in list
    - Picking from front of nearly sorted list
- Can avoid worst-case behavior
  - Select random pivot value

Types of Case Analysis

- Average case
  - Number of steps required for “typical” case
  - Most useful metric in practice
  - Different approaches
    - Average case
    - Expected case

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Approaches to Average Case

- Average case
  - Average over all possible inputs
    - Assumes all inputs have the same probability
  - Example
    - Case 1 = 10 steps, Case 2 = 20 steps
    - Average = 15 steps
- Expected case
  - Weighted average over all possible inputs
    - Based on probability of each input
  - Example
    - Case 1 (90%) = 10 steps, Case 2 (10%) = 20 steps
    - Average = 11 steps

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Average Case Example

- Example problem
  - Average # of comparisons needed to find a number in the (sorted) array A[ ] = {1, 4, 8, 12, 15} using
    1. Linear search
       - Start from beginning, compare elements one at a time
    2. Binary search
       - Start from middle of array at index k, compare element
       - If not element, repeat for top or bottom half of remaining array depending on whether element is smaller or greater than A[k]

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Average Case : Linear Search

- Algorithm
  1. Find # of comparisons needed for each case
     - 1 → 1 comparison (1)
     - 4 → 2 comparisons (1, 4)
     - 8 → 3 comparisons (1, 4, 8)
     - 12 → 4 comparisons (1, 4, 8, 12)
     - 15 → 5 comparisons (1, 4, 8, 12, 15)
  2. Calc average = total # of comparisons / # cases
     - Total # comparisons = 1 + 2 + 3 + 4 + 5 = 15
     - # cases = 5
     - Average = 3 comparisons / number

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Average Case : Binary Search

- Algorithm
  1. Find # of comparisons needed for each case
     - 1 → 3 comparisons (8, 4, 1)
     - 4 → 2 comparisons (8, 4)
     - 8 → 1 comparisons (8)
     - 12 → 2 comparisons (8, 12)
     - 15 → 3 comparisons (8, 12, 15)
  2. Calc average = total # of comparisons / # cases
     - Total # comparisons = 3 + 2 + 1 + 2 + 3 = 11
     - # cases = 5
     - Average = 2.2 comparisons / number

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Average Case Example

- Example problem 2
  - Average # of comparisons needed to find a number in a sorted array A[n] of size n using
    1. Linear search
    2. Binary search
  - For simplicity, we assume elements are stored in A[1] … A[n]

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Average Case : Linear Search

- Algorithm
  1. Find # of comparisons needed for each case
     - A[1] → 1 comparison (A[1])
     - A[2] → 2 comparisons (A[1], A[2])
     - …
     - A[n] → n comparisons (A[1] … A[n])
  2. Calc average = total # of comparisons / # cases
     - Total # comparisons = 1 + 2 + … + n = ½ n^2 + 1
     - # cases = n
     - Average ≈ ½ n comparisons / number

Average Case : Binary Search

- Algorithm
  1. Find # of comparisons needed for each case
     - A[n/2] → 1 comp (A[n/2])
     - A[n/4], A[3n/4] → 2 comps (A[n/2], A[n/4])
     - …
     - A[1], A[3]…A[n] → $\log_2$(n) comparisons (A[n/2], A[n/4], A[n/8]…A[1])
  2. Calc average = total # of comparisons / # cases
     - Total # comparisons = n/2 * $\log_2$(n) + n/4 * $\log_2$(n)–1 + … + 1 = n $\log_2$(n)
     - # cases = n
     - Average ≈ $\log_2$(n) comparisons / number

Amortized Analysis

- Approach
  - Applies to worst-case sequences of operations
  - Finds average running time per operation
  - Example
    - Normal case = 10 steps
    - Every 10^{th} case may require 20 steps
    - Amortized time = 11 steps
- Assumptions
  - Can predict possible sequence of operations
  - Know when worst-case operations are needed
    - Does not require knowledge of probability

Amortization Example

- Adding numbers to end of array of size k
  - If array is full, allocate new array
    - Allocation cost is O(size of new array)
  - Copy over contents of existing array
- Two approaches
  - Non-amortized
    - If array is full, allocate new array of size k+1
  - Amortized
    - If array is full, allocate new array of size 2k
  - Compare their allocation cost

Amortization Example

- Non-amortized approach
  - Allocation cost as table grows from 1..n

Size (k)	1	2	3	4	5	6	7	8
Cost	1	2	3	4	5	6	7	8

  - Total cost ⇒ n(n+1)/2
- Case analysis
  - Best case ⇒ allocation cost = k
  - Worse case ⇒ allocation cost = k
  - Amortized case ⇒ allocation cost = (n+1)/2

Amortization Example

- Amortized approach
  - Allocation cost as table grows from 1..n

Size (k)	1	2	3	4	5	6	7	8
Cost	2	0	4	0	8	0	0	0

  - Total cost ⇒ 2 (n – 1)
- Case analysis
  - Best case ⇒ allocation cost = 0
  - Worse case ⇒ allocation cost = 2(k – 1)
  - Amortized case ⇒ allocation cost = 2
- An individual step might take longer, but faster for any sequence of operations