CMSC 132:
Object-Oriented Programming II

Minimal Spanning Tree Algorithms

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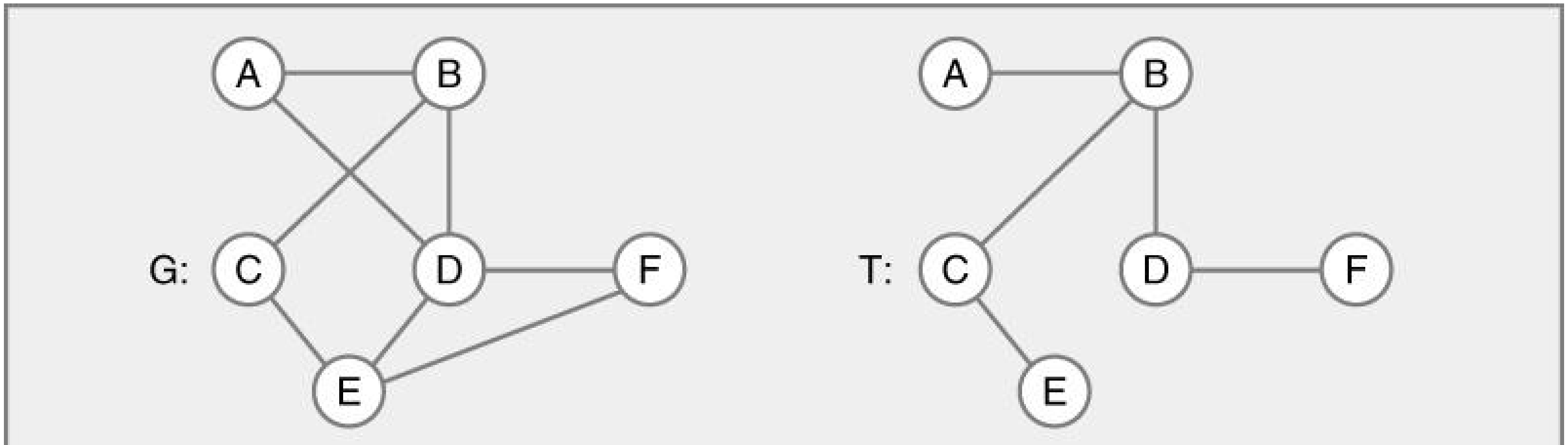
Overview

- Spanning trees
- Minimum spanning tree (MST)
  - Prim's algorithm
  - Kruskal's algorithm
    - Union-Find

Spanning Tree

- Set of edges connecting all nodes in graph
  - need N-1 edges for N nodes
  - no cycles, can be thought of as a tree
- Can build tree during traversal



(a) Graph G (b) Spanning tree T of graph G

Spanning Tree Construction

- **Recursive algorithm**

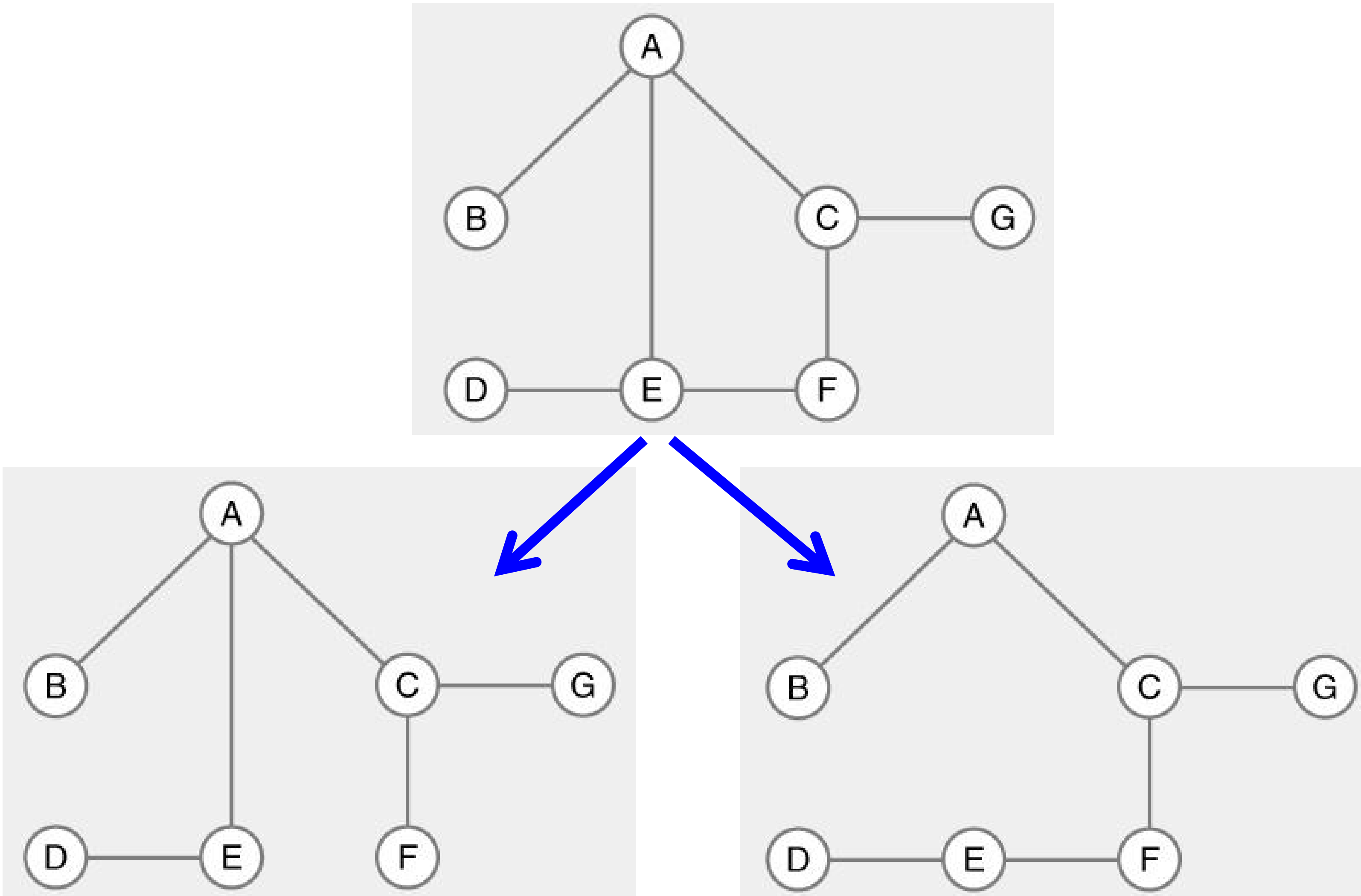
```
Known = { start }
explore ( start );

void explore (Node X) {
    for each successor Y of X
        if (Y is not in Known)
            Parent[Y] = X
            Add Y to Known
            explore(Y)
}
```

Spanning Tree Construction

- **Iterative algorithm**

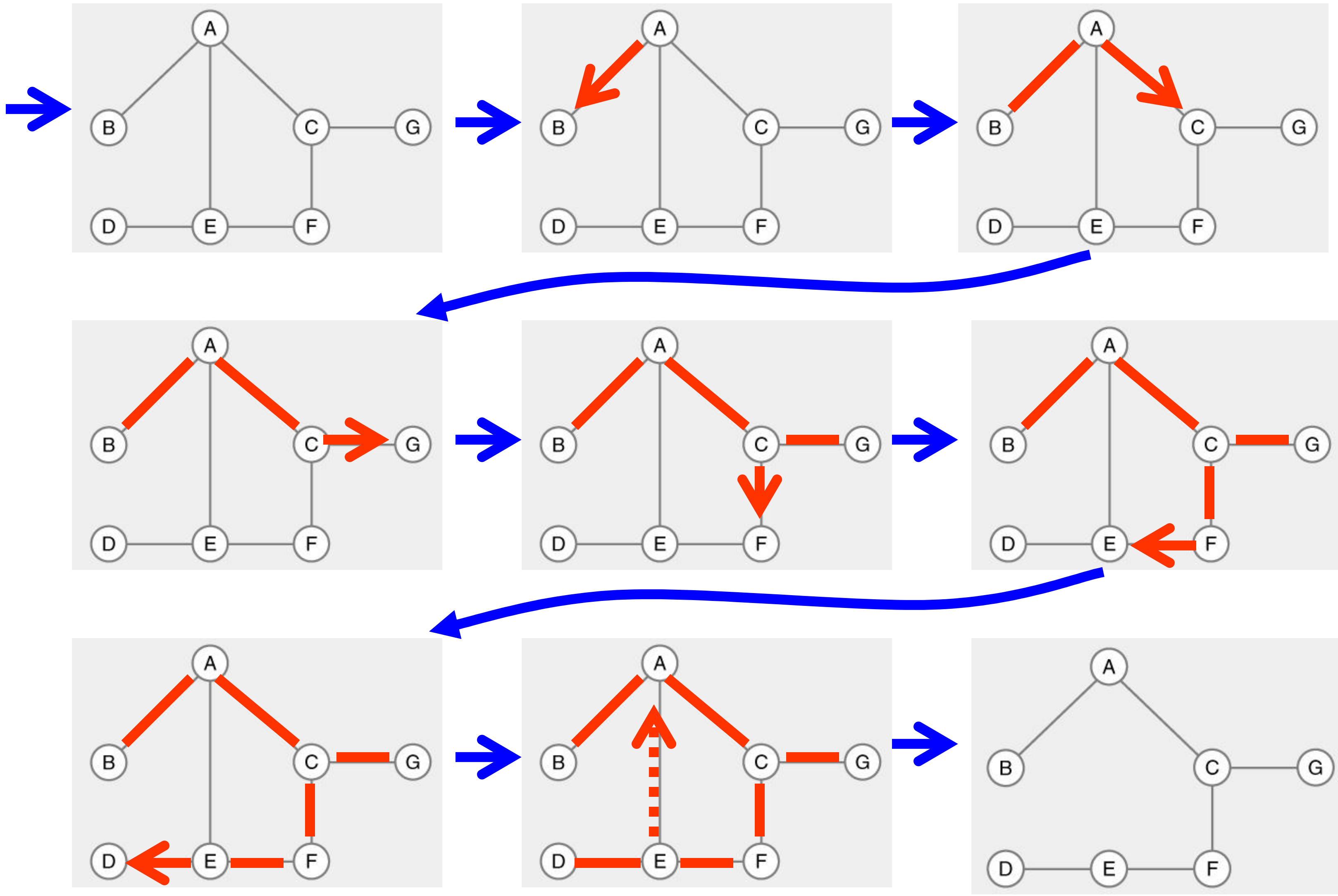
```
Known = { start }
Discovered = { start }
while ( Discovered ≠ ∅ ) {
    take node X out of Discovered
    for each successor Y of X
        if (Y is not in Known)
            Parent[Y] = X
            Add Y to Discovered
            Add Y to Known
}
```

Breadth & Depth First Spanning Trees

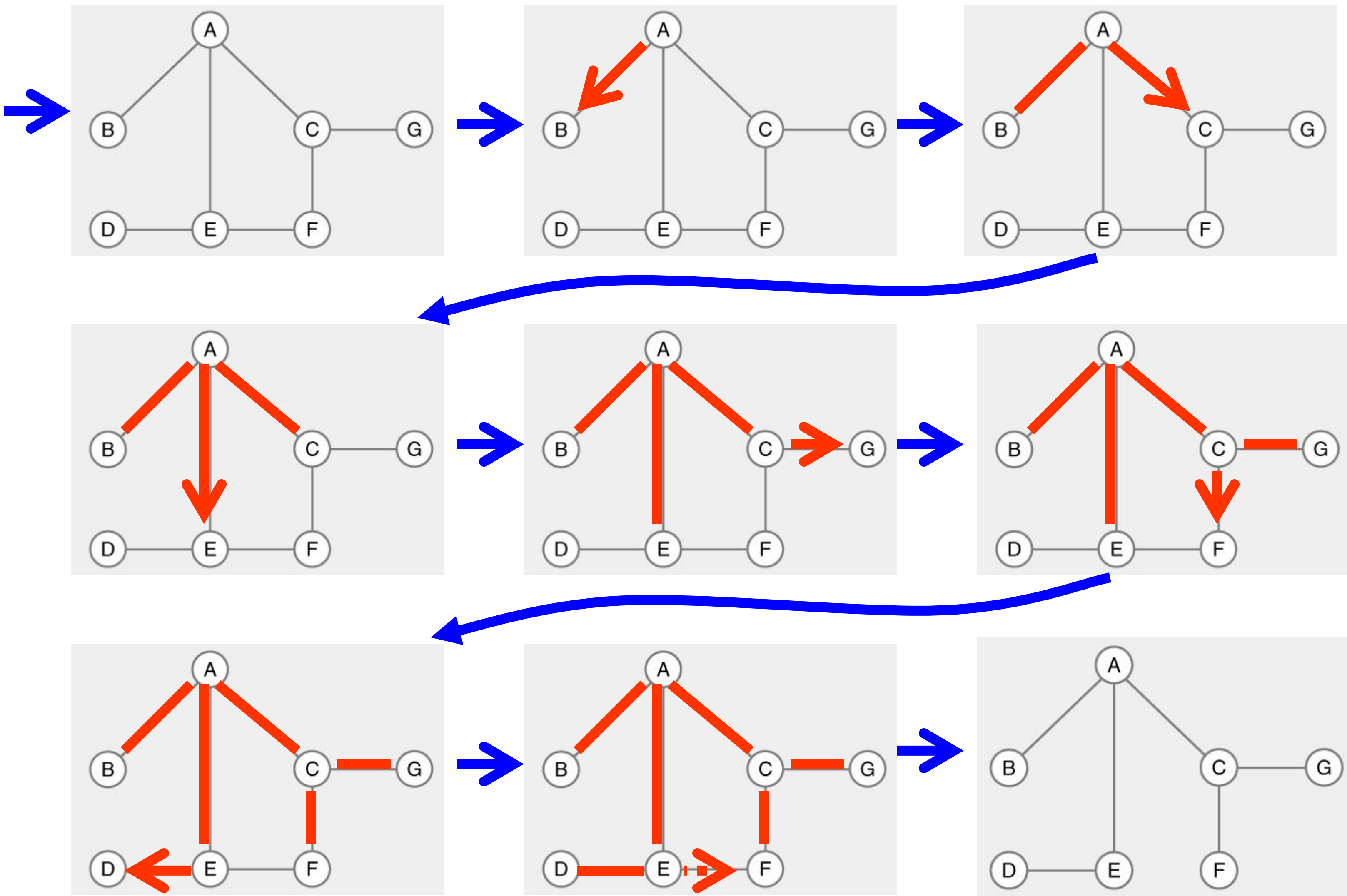


Breadth-first **Depth-first**

Depth-First Spanning Tree Example



Breadth-First Spanning Tree Example

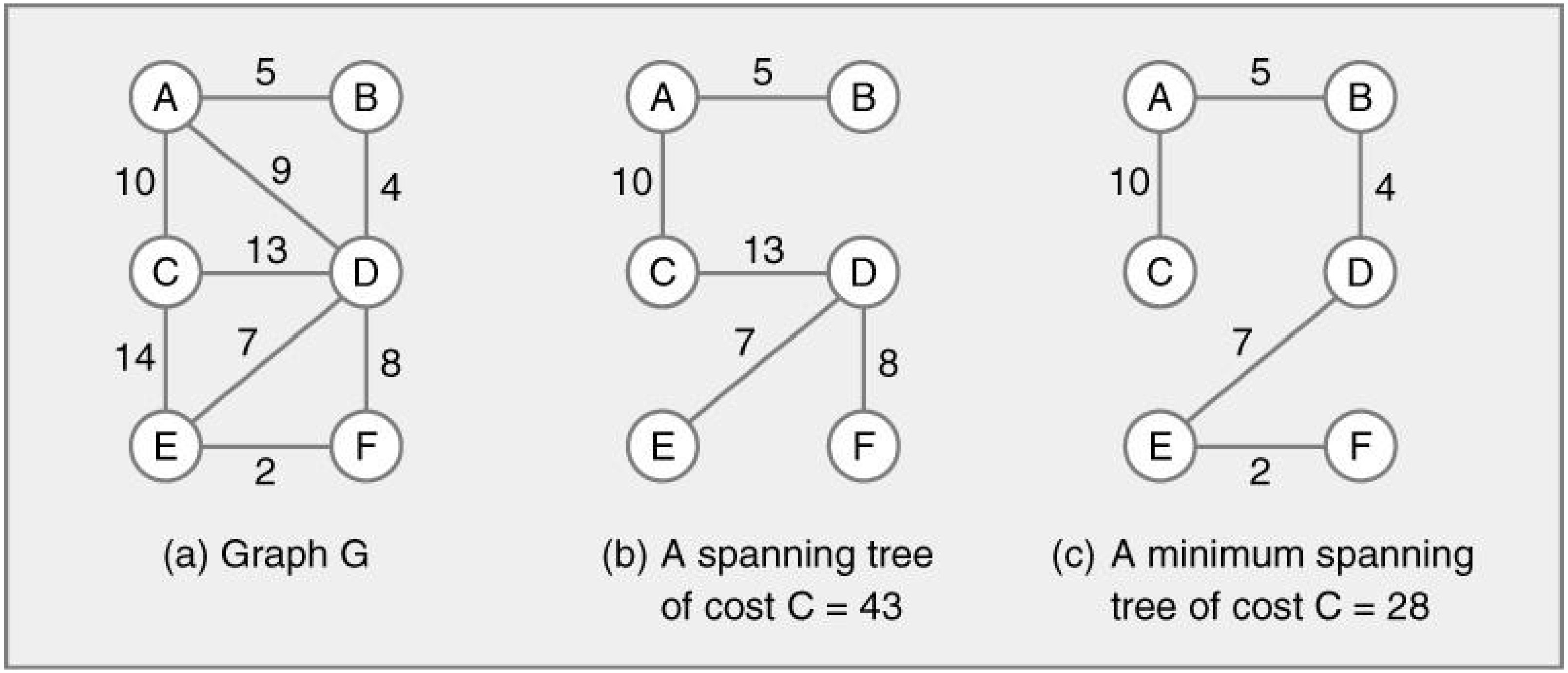


Spanning Tree Construction

- **Many spanning trees possible**
  - **Different breadth-first traversals**
    - **Nodes same distance visited in different order**
  - **Different depth-first traversals**
    - **Neighbors of node visited in different order**
  - **Different traversals yield different spanning trees**

Minimum Spanning Tree (MST)

- **Spanning tree with minimum total edge weight**



Minimum Spanning Tree (MST)

- Possible to have multiple MSTs
  - Different spanning trees with same weight

- Example applications
  - Minimize length of telephone lines for neighborhood
  - Minimize distance of airplane routes serving cities

Algorithms for Finding MST

- Three well known algorithms
  1. Borůvka's algorithm [1926]
     - For constructing efficient electricity network
     - Rediscovered by Sollin in 1960s
  2. Prim's algorithm [1957]
     - First discovered by Vojtěch Jarník in 1930
     - Similar to Djikstra's algorithm
  3. Kruskal's algorithm [1956]
     - By Prof. Clyde Kruskal's uncle

Algorithms for Finding MST

1. **Borůvka's algorithm**
   - Add vertices to MST in parallel
2. **Prim's algorithm**
   - Add vertices to MST
     - One at a time
     - Closest vertex first
3. **Kruskal's algorithm**
   - Add edges to MST
     - One at a time
     - Lightest edge first

Shortest Path – Dijkstra's Algorithm

```
S = ∅
P[ ] = none for all nodes
C[start] = 0, C[ ] = ∞ for all other nodes
while ( not all nodes in S )
    find node K not in S with smallest C[K]
    add K to S
    for each node J not in S  adjacent to K
        if ( C[K] + cost of (K,J) < C[J] )
            C[J] = C[K] + cost of (K,J)
            P[J] = K
```

Optimal solution computed with greedy algorithm

MST – Prim's Algorithm

```
S = ∅
P[ ] = none for all nodes
C[start] = 0, C[ ] = ∞ for all other nodes
while ( not all nodes in S )
   find node K not in S with smallest C[K]
   add K to S
   for each node J not in S  adjacent to K
      if ( /* C[K] + */ cost of (K,J) < C[J] )
         C[J] = /* C[K] + */ cost of (K,J)
         P[J] = K
```

Keeps track of vertex w/ minimal distance to current tree

Optimal solution computed with greedy algorithm

MST – Kruskal's Algorithm

sort edges by weight (from least to most)

tree = ∅

for each edge (X,Y) in order

    if it does not create a cycle

        add (X,Y) to tree

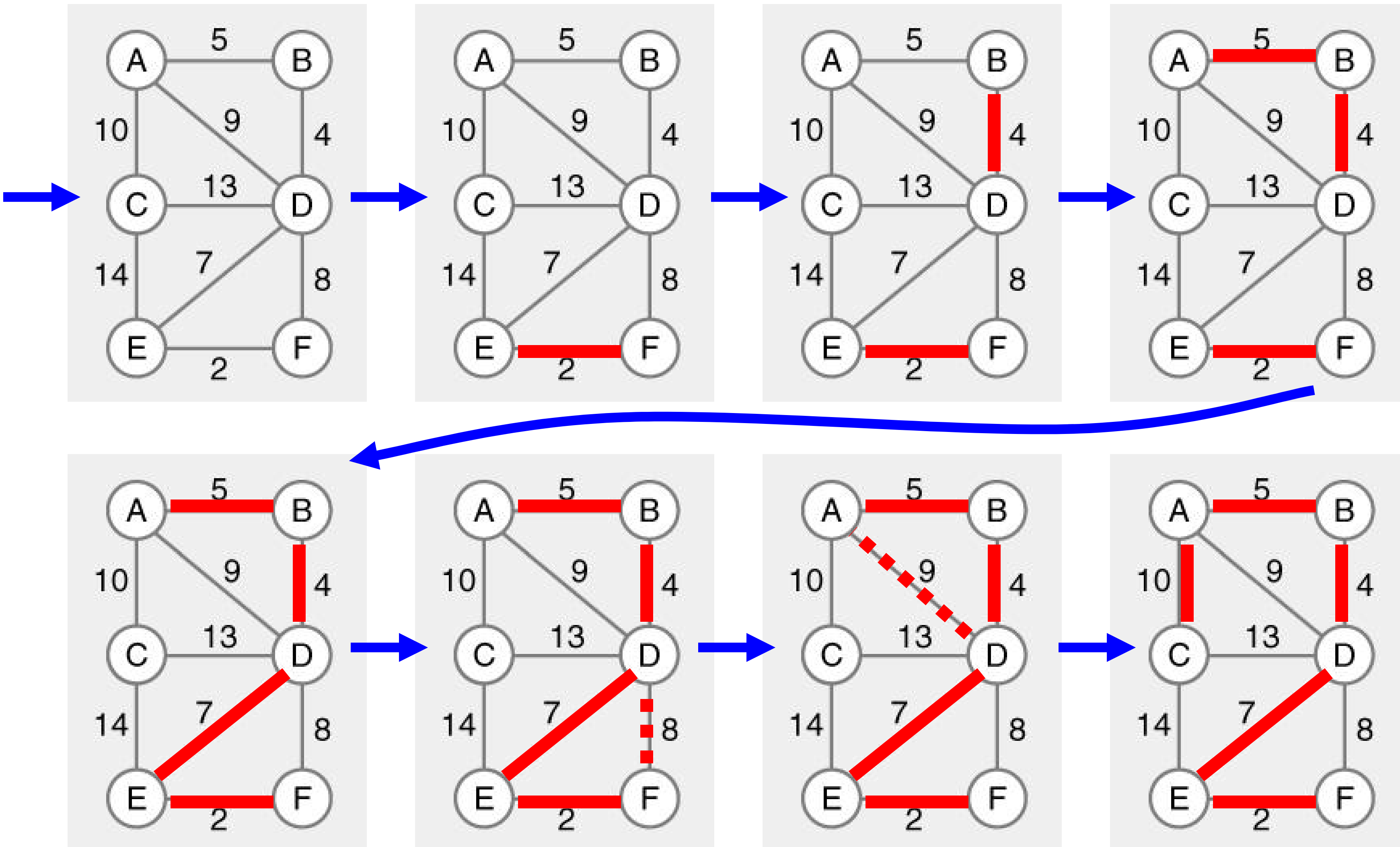
        stop when tree has N–1 edges

Keeps track of

- lightest edge remaining
- whether adding edge to MST creates cycle

Optimal solution computed with greedy algorithm

MST – Kruskal's Algorithm Example



MST – Kruskal’s Algorithm

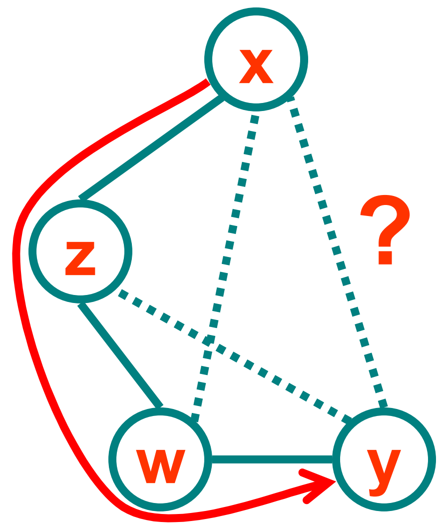
- **When does adding (X,Y) to tree create cycle?**

- **Two approaches to finding cycles**
  1. **Traversal**
  2. **Connected subgraph**

MST – Kruskal's Algorithm

- **Traversal approach**
  - **Traverse tree starting at X**
  - **If we can reach Y, adding (X,Y) would create cycle**

- **Example**
  - **Question**
    - **Add (X,Y) to MST?**
  - **Answer**
    - **No, since can already reach Y from X by traversing MST**

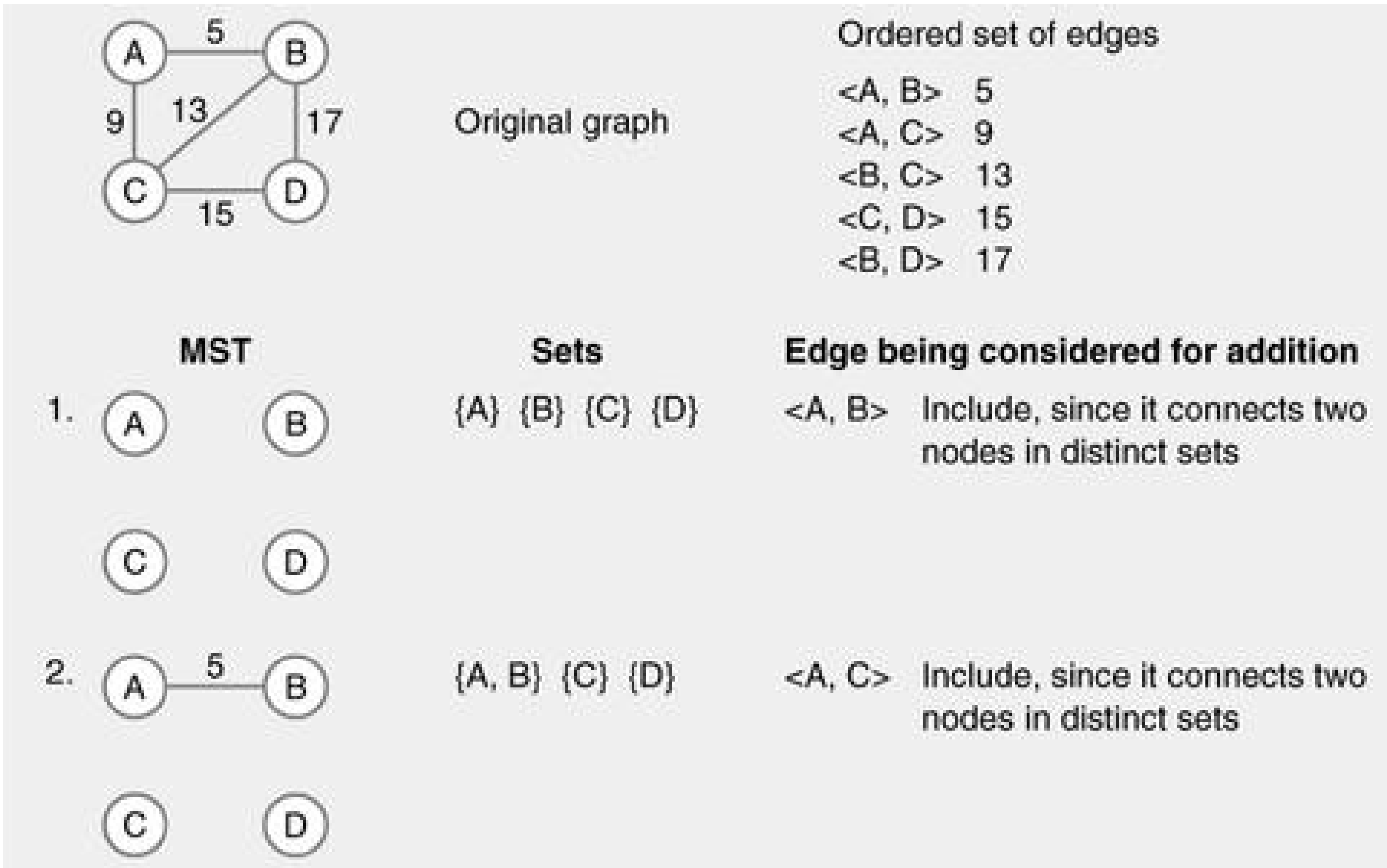


MST – Kruskal's Algorithm

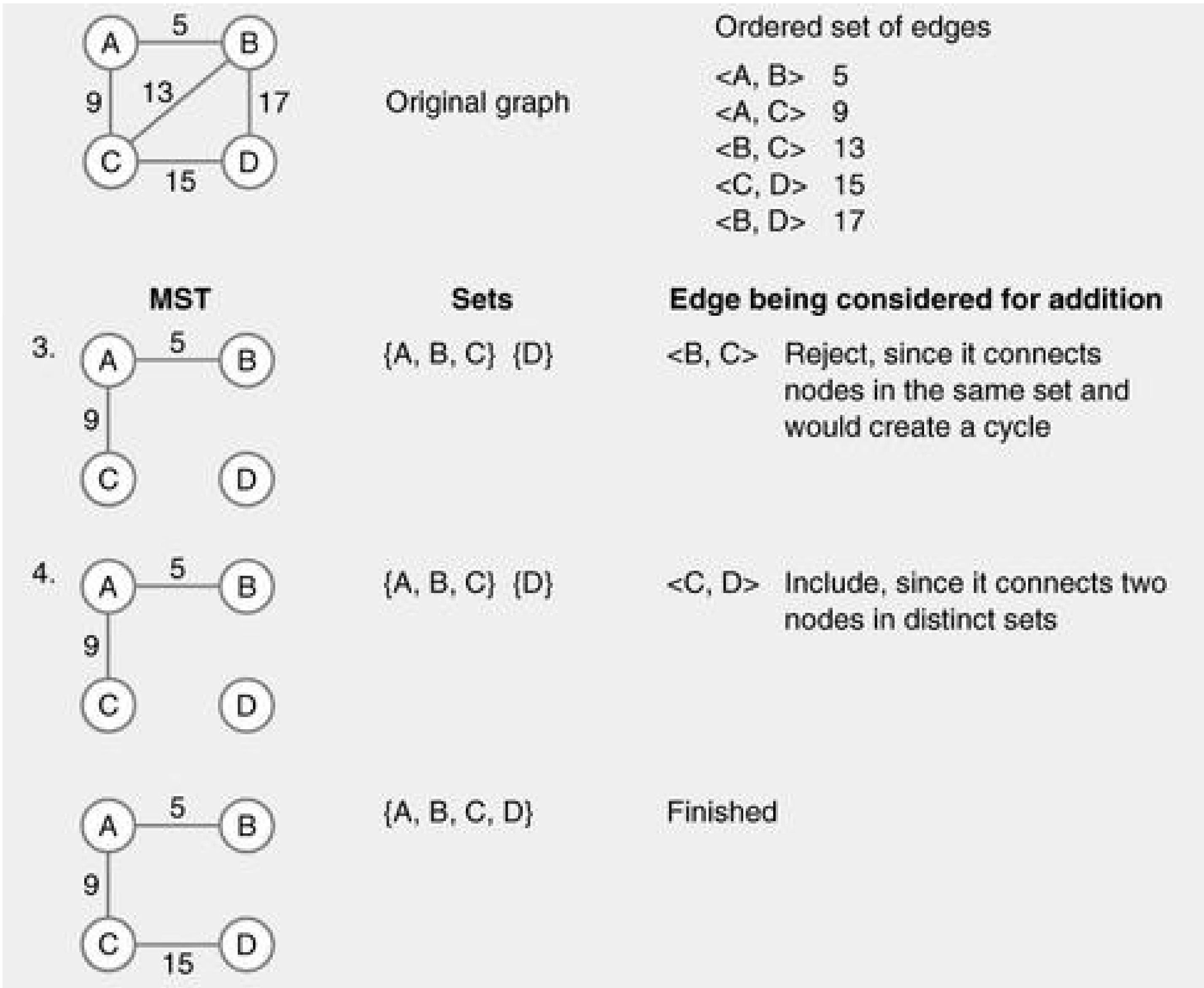
- Connected subgraph approach
  - Maintain set of nodes for each connected subgraph
  - Initialize one connected subgraph for each node
  - If X, Y in same set, adding (X,Y) would create cycle
  - Otherwise
    1. Add edge (X,Y) to spanning tree
    2. Merge sets containing X, Y

- To test set membership
  - Use Union-Find algorithm

MST – Connected Subgraph Example



MST – Connected Subgraph Example



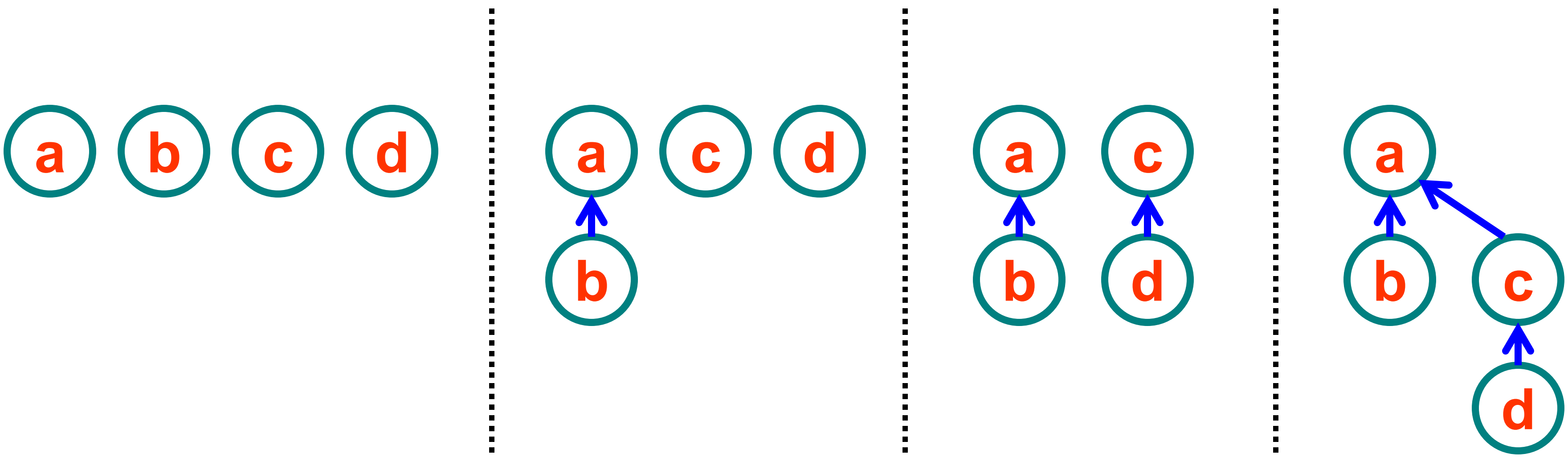
Union-Find Algorithm

- Union-Find
  - Algorithm & data structure
  - Very efficient for testing membership in disjoint sets

- Problem description
  - Start with n nodes, each in different subgraph
  - Support two operations
    - Find – are nodes x & y in same subgraph?
    - Union – merge subgraphs containing x & y

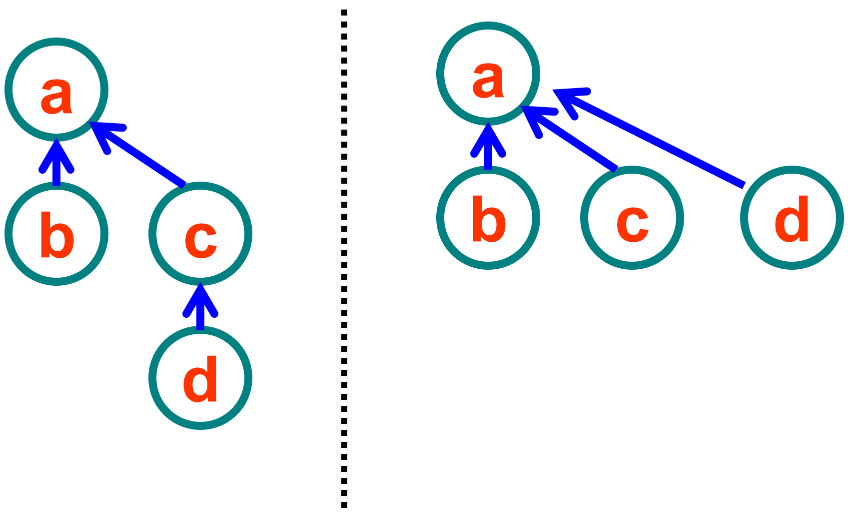
Union-Find Algorithm

- **Basic approach**
  - **Each node has a parent pointer**
  - **Find – follow parent pointer(s) to root of tree**
  - **Union – point root of 1st tree to root of 2nd tree**
- **Example**
  - **Union(a, b) ; union(c , d); union(b, d)**



Union-Find Algorithm

- **Path compression**
  - Speeds up future Find( ) operations
    1. Follow parent pointer(s) to root of tree
    2. Update all nodes along path to point to root
- **Example**
  - Find(d)



So how fast is Union-Find?

Ackermann's Function

- Function

```
int A(x,y) {
    if (x == 0)
      return y+1;
    if (y == 0)
      return A(x – 1, 1);
    return A(x – 1, A(x, y – 1));
}
```

- A( ) grows fast
  - A(2,2) = 7
  - A(3,3) = 61
  - A(4,2) = $2^{65536} - 3$
  - A(4,3) = $2^{2^{65536}} - 3$
  - A(4,4) = $2^{2^{2^{65536}}} - 3$

Inverse Ackermann's Function

- Definition
  - $\alpha(n)$ is the inverse Ackermann's function
  - $\alpha(n)$ = the smallest k such that $A(k,k) \geq n$
- Fun fact
  - α(number of atoms in universe) = 4
- Union-find
  - A sequence of n operations requires $O(n\ \alpha(n))$ time
  - Practically speaking, indistinguishable from $O(n)$

Graph Summary

- Graph data structure
  - Very useful in practice
  - Different representations
- Many graph algorithms
  - Traversal
  - Shortest path
  - Minimum spanning tree

Algorithms / Data Structures

- Introduction to data structures in 132
  - Lists, Trees, Graphs, Sets / Maps
- Much more to learn in future courses
  - 351 – Introduction to Algorithms
    - Dynamic programming, recurrences, reductions, NP-completeness…
  - 420 – Data Structures
    - Balanced trees, quadtrees, k-d trees…
  - 451 – Design and Analysis of Computer Algorithms
    - Correctness proofs, analyzing complexity…