CMSC 132: Object-Oriented Programming II

Minimal Spanning Tree Algorithms

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Overview

- Spanning trees
- Minimum spanning tree (MST)
  - Prim’s algorithm
  - Kruskal’s algorithm
  - Union-Find
Spanning Tree

- Set of edges connecting all nodes in graph
  - need N-1 edges for N nodes
  - no cycles, can be thought of as a tree

- Can build tree during traversal

(a) Graph G
(b) Spanning tree T of graph G
Spanning Tree Construction

Recursive algorithm

Known = { start }
explore ( start );

void explore (Node X) {
    for each successor Y of X
        if (Y is not in Known)
            Parent[Y] = X
            Add Y to Known
            explore(Y)
}

Spanning Tree Construction

Iterative algorithm

Known = { start }
Discovered = { start }
while ( Discovered ≠ ∅ ) {
    take node X out of Discovered
    for each successor Y of X
        if (Y is not in Known)
            Parent[Y] = X
            Add Y to Discovered
            Add Y to Known
}

Breadth & Depth First Spanning Trees

Breadth-first

Depth-first
Depth-First Spanning Tree Example
Breadth-First Spanning Tree Example
Spanning Tree Construction

Many spanning trees possible

- Different breadth-first traversals
  - Nodes same distance visited in different order
- Different depth-first traversals
  - Neighbors of node visited in different order
- Different traversals yield different spanning trees
Minimum Spanning Tree (MST)

Spanning tree with minimum total edge weight

(a) Graph G
(b) A spanning tree of cost $C = 43$
(c) A minimum spanning tree of cost $C = 28$
Minimum Spanning Tree (MST)

- Possible to have multiple MSTs
  - Different spanning trees with same weight

- Example applications
  - Minimize length of telephone lines for neighborhood
  - Minimize distance of airplane routes serving cities
Algorithms for Finding MST

Three well known algorithms

1. **Borůvka’s algorithm** [1926]
   - For constructing efficient electricity network
   - Rediscovered by Sollin in 1960s

2. **Prim’s algorithm** [1957]
   - First discovered by Vojtěch Jarník in 1930
   - Similar to Djikstra’s algorithm

3. **Kruskal’s algorithm** [1956]
   - By Prof. Clyde Kruskal’s uncle
Algorithms for Finding MST

1. Borůvka’s algorithm
   - Add vertices to MST in parallel

2. Prim’s algorithm
   - Add vertices to MST
     - One at a time
     - Closest vertex first

3. Kruskal’s algorithm
   - Add edges to MST
     - One at a time
     - Lightest edge first
Shortest Path – Dijkstra’s Algorithm

$S = \emptyset$

$P[\ ] = \text{none for all nodes}$

$C[\text{start}] = 0$, $C[\ ] = \infty$ for all other nodes

while ( not all nodes in $S$ )

    find node $K$ not in $S$ with smallest $C[K]$

    add $K$ to $S$

    for each node $J$ not in $S$ adjacent to $K$

        if ( $C[K] + \text{cost of (K,J)} < C[J]$ )

            $C[J] = C[K] + \text{cost of (K,J)}$

            $P[J] = K$

Optimal solution computed with greedy algorithm
MST – Prim’s Algorithm

S = ∅
P[ ] = none for all nodes  
C[start] = 0, C[ ] = ∞ for all other nodes
while ( not all nodes in S )
    find node K not in S with smallest C[K]
    add K to S
    for each node J not in S adjacent to K
        if ( /* C[K] + */ cost of (K,J) < C[J] )
            C[J] = /* C[K] + */ cost of (K,J)
            P[J] = K

Keeps track of vertex w/ minimal distance to current tree
Optimal solution computed with greedy algorithm
MST – Kruskal’s Algorithm

sort edges by weight (from least to most)

tree = ∅

for each edge (X,Y) in order

    if it does not create a cycle
        add (X,Y) to tree
    stop when tree has N–1 edges

Keeps track of

- lightest edge remaining
- whether adding edge to MST creates cycle

Optimal solution computed with greedy algorithm
MST – Kruskal’s Algorithm Example
MST – Kruskal’s Algorithm

- When does adding (X,Y) to tree create cycle?

- Two approaches to finding cycles
  1. Traversal
  2. Connected subgraph
MST – Kruskal’s Algorithm

Traversal approach

- Traverse tree starting at X
- If we can reach Y, adding (X,Y) would create cycle

Example

- Question
  - Add (X,Y) to MST?
- Answer
  - No, since can already reach Y from X by traversing MST
MST – Kruskal’s Algorithm

Connected subgraph approach

- Maintain set of nodes for each connected subgraph
- Initialize one connected subgraph for each node
- If X, Y in same set, adding (X,Y) would create cycle
- Otherwise
  1. Add edge (X,Y) to spanning tree
  2. Merge sets containing X, Y

To test set membership

- Use Union-Find algorithm
MST – Connected Subgraph Example

Original graph

Ordered set of edges

<\text{A, B}> 5
<\text{A, C}> 9
<\text{B, C}> 13
<\text{C, D}> 15
<\text{B, D}> 17

MST

Sets

{\text{A}} \quad \{\text{B}} \quad \{\text{C}} \quad \{\text{D}}

_edge being considered for addition_

<\text{A, B}> 
Include, since it connects two nodes in distinct sets

<\text{A, C}> 
Include, since it connects two nodes in distinct sets
MST – Connected Subgraph Example

Original graph

Ordered set of edges

1. <A, B> 5
2. <A, C> 9
3. <B, C> 13
4. <C, D> 15
5. <B, D> 17

MST

Sets

3. MST

{A, B, C} {D}

Edge being considered for addition

1. <B, C> Reject, since it connects nodes in the same set and would create a cycle
2. <C, D> Include, since it connects two nodes in distinct sets

Finished
Union-Find Algorithm

Union-Find
- Algorithm & data structure
- Very efficient for testing membership in disjoint sets

Problem description
- Start with n nodes, each in different subgraph
- Support two operations
  - Find – are nodes x & y in same subgraph?
  - Union – merge subgraphs containing x & y
Union-Find Algorithm

**Basic approach**
- Each node has a parent pointer
- Find – follow parent pointer(s) to root of tree
- Union – point root of 1\textsuperscript{st} tree to root of 2\textsuperscript{nd} tree

**Example**
- Union( a, b ) ; union( c , d); union( b, d)
Union-Find Algorithm

Path compression

- Speeds up future Find( ) operations
  1. Follow parent pointer(s) to root of tree
  2. Update all nodes along path to point to root

Example

- Find(d)

So how fast is Union-Find?
Ackermann’s Function

Function

```c
int A(x, y) {
    if (x == 0)
        return y + 1;
    if (y == 0)
        return A(x - 1, 1);
    return A(x - 1, A(x, y - 1));
}
```

A( ) grows fast

- A(2, 2) = 7
- A(3, 3) = 61
- A(4, 2) = 2^{65536} - 3
- A(4, 3) = 2^{2^{65536}} - 3
- A(4, 4) = 2^{2^{2^{65536}}} - 3
Inverse Ackermann’s Function

Definition

\( \alpha(n) \) is the inverse Ackermann’s function

\( \alpha(n) = \text{the smallest } k \text{ such that } A(k,k) \geq n \)

Fun fact

\( \alpha(\text{number of atoms in universe}) = 4 \)

Union-find

A sequence of \( n \) operations requires \( O(n \alpha(n)) \) time

Practically speaking, indistinguishable from \( O(n) \)
Graph Summary

- Graph data structure
  - Very useful in practice
  - Different representations

- Many graph algorithms
  - Traversal
  - Shortest path
  - Minimum spanning tree
Algorithms / Data Structures

- Introduction to data structures in 132
  - Lists, Trees, Graphs, Sets / Maps
- Much more to learn in future courses
  - 351 – Introduction to Algorithms
    - Dynamic programming, recurrences, reductions, NP-completeness…
  - 420 – Data Structures
    - Balanced trees, quadtrees, k-d trees…
  - 451 – Design and Analysis of Computer Algorithms
    - Correctness proofs, analyzing complexity…