CMSC 132: Object-Oriented Programming II

Algorithmic Complexity I

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Algorithm Efficiency

**Efficiency**

- Amount of resources used by algorithm
  - Time, space

**Measuring efficiency**

- Benchmarking
- Asymptotic analysis
Benchmarking

Approach
- Pick some desired inputs
- Actually run implementation of algorithm
- Measure time & space needed

Industry benchmarks
- SPEC – CPU performance
- MySQL – Database applications
- WinStone – Windows PC applications
- MediaBench – Multimedia applications
- Linpack – Numerical scientific applications
Benchmarking

Advantages
- Precise information for given configuration
  - Implementation, hardware, inputs

Disadvantages
- Affected by configuration
  - Data sets (often too small)
    - a dataset that was the right size 3 years ago is likely too small now
  - Hardware
  - Software
  - Affected by special cases (biased inputs)
  - Does not measure intrinsic efficiency
Asymptotic Analysis

Approach
- Mathematically analyze efficiency
- Calculate time as function of input size $n$
  - $T \equiv O( f(n) )$
  - $T$ is on the order of $f(n)$
  - “Big O” notation

Advantages
- Measures intrinsic efficiency
- Dominates efficiency for large input sizes
- Programming language, compiler, processor irrelevant
Search Example

Number guessing game

- Pick a number between 1…n
- Guess a number
- Answer “correct”, “too high”, “too low”
- Repeat guesses until correct number guessed
Linear Search Algorithm

**Algorithm**
- Guess number = 1
- If incorrect, increment guess by 1
- Repeat until correct

**Example**
- Given number between 1…100
- Pick 20
- Guess sequence = 1, 2, 3, 4 … 20
- Required 20 guesses
Linear Search Algorithm

Analysis of # of guesses needed for 1…n

- If number = 1, requires 1 guess
- If number = n, requires n guesses
- On average, needs n/2 guesses
- Time = O( n ) = Linear time
Binary Search Algorithm

**Algorithm**

- Set low and high to be lowest and highest possible value
- Guess middle = (low+high)/2
- If too large, set high = middle-1
- If too small, set low = middle+1
- Repeat until guess correct
Binary Search Algorithm

Example

Given number between 1…100
secret number we are trying is find is 20

Guesses

- low = 1, high = 100, guess 50, Answer = too large
- low = 1, high = 49, guess 25, Answer = too large
- low = 1, high = 24, guess 12, Answer = too small
- low = 13, high = 24, guess 18, Answer = too small
- low = 19, high = 24, guess 21, Answer = too large
- low = 19, high = 20, guess 19, Answer = too small
- low = 20, high = 20, guess 20, Answer = correct

Required 7 guesses
Binary Search Algorithm

Analysis of # of guesses needed for 1...n

- If number = n/2, requires 1 guess
- If number = 1, requires $\log_2(n)$ guesses
- If number = n, requires $\log_2(n)$ guesses
- On average, needs $\log_2(n)$ guesses
- Time = $O(\log_2(n)) = O(\log(n)) = \text{Log time}$
Search Comparison

For number between 1…100

- Simple algorithm = 50 steps
- Binary search algorithm = $\log_2(n) = 7$ steps

For number between 1…100,000

- Simple algorithm = 50,000 steps
- Binary search algorithm = $\log_2(n)$ (about 17 steps)

Binary search is much more efficient!
### Asymptotic Complexity

**Comparing two linear functions**

<table>
<thead>
<tr>
<th>Size</th>
<th>Running Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n/2</td>
</tr>
<tr>
<td>64</td>
<td>32</td>
</tr>
<tr>
<td>128</td>
<td>64</td>
</tr>
<tr>
<td>256</td>
<td>128</td>
</tr>
<tr>
<td>512</td>
<td>256</td>
</tr>
</tbody>
</table>
Asymptotic Complexity

Comparing two functions
- \( n/2 \) and \( 4n+3 \) behave similarly
- Run time roughly doubles as input size doubles
- Run time increases linearly with input size

For large values of \( n \)
- \( \frac{\text{Time}(2n)}{\text{Time}(n)} \) approaches exactly 2

Both are \( O(n) \) programs
Asymptotic Complexity

Comparing two log functions

<table>
<thead>
<tr>
<th>Size</th>
<th>Running Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \log_2(n) )</td>
</tr>
<tr>
<td>64</td>
<td>6</td>
</tr>
<tr>
<td>128</td>
<td>7</td>
</tr>
<tr>
<td>256</td>
<td>8</td>
</tr>
<tr>
<td>512</td>
<td>9</td>
</tr>
</tbody>
</table>
Asymptotic Complexity

- Comparing two functions
  - \( \log_2(n) \) and \( 5 \times \log_2(n) + 3 \) behave similarly
  - Run time roughly increases by constant as input size doubles
  - Run time increases logarithmically with input size
- For large values of \( n \)
  - \( \text{Time}(2n) - \text{Time}(n) \) approaches constant
  - Base of logarithm does not matter
    - Simply a multiplicative factor
      \[ \log_a N = (\log_b N) / (\log_b a) \]
  - Both are \( O(\log(n)) \) programs
### Asymptotic Complexity

#### Comparing two quadratic functions

<table>
<thead>
<tr>
<th>Size</th>
<th>Running Time</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n²</td>
<td>2 n² + 8</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>40</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>132</td>
</tr>
<tr>
<td>16</td>
<td>256</td>
<td>520</td>
</tr>
</tbody>
</table>
Asymptotic Complexity

- Comparing two functions
  - $n^2$ and $2n^2 + 8$ behave similarly
  - Run time roughly increases by 4 as input size doubles
  - Run time increases quadratically with input size
- For large values of $n$
  - $\frac{\text{Time}(2n)}{\text{Time}(n)}$ approaches 4
- Both are $O(n^2)$ programs
Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
  - For sufficiently large input size
  - Intrinsic efficiency of algorithm for large inputs

![Graph showing # steps vs input size with O(...) and f(n) curves]
Formal Definition of Big-O

Function $f(n)$ is $O(g(n))$ if

- For some positive constants $M, N_0$
- $M \times g(n) \geq f(n)$, for all $n \geq N_0$

Intuitively

- For some coefficient $M$ & all data sizes $\geq N_0$
  - $M \times g(n)$ is always greater than $f(n)$
Big-O Examples

5n + 1000 ⇒ O(n)

- Select M = 6, N₀ = 1000
- For n ≥ 1000
  - 6n ≥ 5n + 1000 is always true
- Example ⇒ for n = 1000
  - 6000 ≥ 5000 + 1000
Big-O Examples

\[ 2n^2 + 10n + 1000 \Rightarrow O(n^2) \]

- Select \( M = 4, N_0 = 100 \)
- For \( n \geq 100 \)
  - \( 4n^2 \geq 2n^2 + 10n + 1000 \) is always true
- Example \( \Rightarrow \) for \( n = 100 \)
  - \( 40000 \geq 20000 + 1000 + 1000 \)
Observations

For large values of n

- Any $O(\log(n))$ algorithm is faster than $O(n)$
- Any $O(n)$ algorithm is faster than $O(n^2)$

Asymptotic complexity is fundamental measure of efficiency
# Asymptotic Complexity Categories

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Name</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(1)</td>
<td>Constant</td>
<td>Array access</td>
</tr>
<tr>
<td>O(log(n))</td>
<td>Logarithmic</td>
<td>Binary search</td>
</tr>
<tr>
<td>O(n)</td>
<td>Linear</td>
<td>Largest element</td>
</tr>
<tr>
<td>O(n log(n))</td>
<td>N log N</td>
<td>Optimal sort</td>
</tr>
<tr>
<td>O(n^2)</td>
<td>Quadratic</td>
<td>2D Matrix addition</td>
</tr>
<tr>
<td>O(n^3)</td>
<td>Cubic</td>
<td>2D Matrix multiply</td>
</tr>
<tr>
<td>O(n^k)</td>
<td>Polynomial</td>
<td>Linear programming</td>
</tr>
<tr>
<td>O(k^n)</td>
<td>Exponential</td>
<td>Integer programming</td>
</tr>
<tr>
<td>O(n!)</td>
<td>Factorial</td>
<td>Brute-force search TSP</td>
</tr>
<tr>
<td>O(n^n)</td>
<td>N to the N</td>
<td></td>
</tr>
</tbody>
</table>

From smallest to largest

For size $n$, constant $k > 1$
Comparison of Complexity

A Comparison of Orders

\[
\begin{align*}
  f(x) &= n \\
  f(x) &= \frac{1}{2} n^2 \\
  f(x) &= n^3
\end{align*}
\]
Complexity Category Example

![Graph showing complexity categories](image-url)

- **2^n**
- **n^2**
- **nlog(n)**
- **n**
- **log(n)**

The graph illustrates the number of solution steps as a function of problem size for different complexity categories.
### Complexity Category Example

<table>
<thead>
<tr>
<th>Problem Size</th>
<th># of Solution Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2^n$</td>
</tr>
<tr>
<td>10</td>
<td>$n^2$</td>
</tr>
<tr>
<td>100</td>
<td>$n \log(n)$</td>
</tr>
<tr>
<td>1000</td>
<td>$n$</td>
</tr>
<tr>
<td>2</td>
<td>$\log(n)$</td>
</tr>
</tbody>
</table>

![Graph showing the growth of solution steps](image)

- $2^n$ (purple crosses)
- $n^2$ (blue crosses)
- $n \log(n)$ (red triangles)
- $n$ (pink squares)
- $\log(n)$ (blue diamonds)

The graph plots the number of solution steps against the problem size, illustrating the growth patterns for different complexity categories.
Calculating Asymptotic Complexity

- As \( n \) increases
  - Highest complexity term dominates
  - Can ignore lower complexity terms

Examples

- \( 2n + 100 \Rightarrow O(n) \)
- \( n \log(n) + 10n \Rightarrow O(n\log(n)) \)
- \( \frac{1}{2}n^2 + 100n \Rightarrow O(n^2) \)
- \( n^3 + 100n^2 \Rightarrow O(n^3) \)
- \( \frac{1}{100}2^n + 100n^4 \Rightarrow O(2^n) \)
Complexity Examples

2n + 100 ⇒ O(n)
Complexity Examples

$\frac{1}{2} n \log(n) + 10 n \Rightarrow O(n\log(n))$
Complexity Examples

$\frac{1}{2} n^2 + 100 n \Rightarrow O(n^2)$
Complexity Examples

\[
\frac{1}{100} 2^n + 100 n^4 \Rightarrow O(2^n)
\]
Types of Case Analysis

Can analyze different types (cases) of algorithm behavior

Types of analysis

- Best case
- Worst case
- Average case
- Amortized
Types of Case Analysis

Best case

- Smallest number of steps required
- Not very useful
- Example ⇒ Find item in first place checked
Types of Case Analysis

Worst case

- Largest number of steps required
- Useful for upper bound on worst performance
  - Real-time applications (e.g., multimedia)
  - Quality of service guarantee
- Example ⇒ Find item in last place checked
**Quicksort Example**

- **Quicksort**
  - One of the fastest comparison sorts
  - Frequently used in practice

- **Quicksort algorithm**
  - Pick *pivot* value from list
  - Partition list into values smaller & bigger than pivot
  - Recursively sort both lists
Quicksort Example

Quicksort properties

- Average case = $O(n\log(n))$
- Worst case = $O(n^2)$
  - Pivot $\approx$ smallest / largest value in list
  - Picking from front of nearly sorted list

Can avoid worst-case behavior

- Select random pivot value
Types of Case Analysis

Average case

- Number of steps required for “typical” case
- Most useful metric in practice
- Different approaches
  - Average case
  - Expected case
Approaches to Average Case

- **Average case**
  - Average over all possible inputs
    - Assumes all inputs have the same probability
  - **Example**
    - Case 1 = 10 steps, Case 2 = 20 steps
    - Average = 15 steps

- **Expected case**
  - Weighted average over all possible inputs
    - Based on probability of each input
  - **Example**
    - Case 1 (90%) = 10 steps, Case 2 (10%) = 20 steps
    - Average = 11 steps
Average Case Example

Example problem

Average # of comparisons needed to find a number in the (sorted) array $A[] = \{1, 4, 8, 12, 15\}$ using

- **Linear search**
  - Start from beginning, compare elements one at a time

- **Binary search**
  - Start from middle of array at index $k$, compare element
  - If not element, repeat for top or bottom half of remaining array depending on whether element is smaller or greater than $A[k]$
Average Case: Linear Search

Algorithm

Find # of comparisons needed for each case

1 → 1 comparison (1)
4 → 2 comparisons (1, 4)
8 → 3 comparisons (1, 4, 8)
12 → 4 comparisons (1, 4, 8, 12)
15 → 5 comparisons (1, 4, 8, 12, 15)

Calc average = total # of comparisons / # cases

Total # comparisons = 1 + 2 + 3 + 4 + 5 = 15
# cases = 5
Average = 3 comparisons / number
Average Case: Binary Search

Algorithm

Find # of comparisons needed for each case

- 1 → 3 comparisons (8, 4, 1)
- 4 → 2 comparisons (8, 4)
- 8 → 1 comparisons (8)
- 12 → 2 comparisons (8, 12)
- 15 → 3 comparisons (8, 12, 15)

Calc average = total # of comparisons / # cases

- Total # comparisons = 3 + 2 + 1 + 2 + 3 = 11
- # cases = 5
- Average = 2.2 comparisons / number
Average Case Example

Example problem 2

Average # of comparisons needed to find a number in a sorted array A[n] of size n using
- Linear search
- Binary search

For simplicity, we assume elements are stored in A[1] ... A[n]
Average Case : Linear Search

**Algorithm**

- Find # of comparisons needed for each case
  - ...

- Calc average = total # of comparisons / # cases
  - Total # comparisons = 1 + 2 + ... + n = ½ n^2 + 1
  - # cases = n
  - Average ≈ ½ n comparisons / number
Average Case : Binary Search

Algorithm

Find # of comparisons needed for each case

- $A[n/2] \rightarrow 1$ comp \quad (A[n/2])
- $A[n/4], A[3n/4] \rightarrow 2$ comps \quad (A[n/2], A[n/4])
- ...
- $A[1], A[3]...A[n] \rightarrow \log_2(n)$ comparisons
- \quad (A[n/2], A[n/4], A[n/8]...A[1])

Calc average = total # of comparisons / # cases

- Total # comparisons = $n/2 \times \log_2(n)$ + $n/4 \times \log_2(n)-1 + ... + 1 = n \log_2(n)$
- # cases = $n$
- Average $\approx \log_2(n)$ comparisons / number
Given an array $a$ of integers

find the subrange that has the maximum sum

e.g., find low, high that maximizes

$$a[\text{low}] + a[\text{low}+1] + \ldots + a[\text{high}]$$

only non empty ranges ($\text{low} \leq \text{high}$)

If $a$ contained only nonnegative integers, would be $\text{low} = 0$, $\text{high} = a.\text{length} - 1$

but $a$ can contain negative numbers

Can assume that arithmetic overflow isn't an issue
One solution

```java
public static int findBestRange(int[] a) {
    int bestSum = a[0];
    for (int low = 0; low < a.length; low++) {
        for (int high = low; high < a.length; high++) {
            int sum = 0;
            for (int i = low; i <= high; i++) sum += a[i];
            if (bestSum < sum)
                bestSum = sum;
        }
    }
    return bestSum;
}

// What is the complexity of the algorithm used here?
```
Can you find a better algorithm?