CMSC 132: Object-Oriented Programming II

Algorithmic Complexity II

Department of Computer Science
University of Maryland, College Park
Overview

- Critical sections
- Comparing complexity
- Types of complexity analysis
Analyzing Algorithms

Goal
- Find asymptotic complexity of algorithm

Approach
- Ignore less frequently executed parts of algorithm
- Find critical section of algorithm
- Determine how many times critical section is executed as function of problem size
Critical Section of Algorithm

- Heart of algorithm
- Dominates overall execution time

Characteristics
- Operation central to functioning of program
- Contained inside deeply nested loops
- Executed as often as any other part of algorithm

Sources
- Loops
- Recursion
Critical Section Example 1

Code (for input size $n$)

1. A
2. for (int i = 0; i < n; i++)
3. B
4. C

critical section

Code execution

- A $\Rightarrow$ once
- B $\Rightarrow$ n times
- C $\Rightarrow$ once

Time $\Rightarrow 1 + n + 1 = O(n)$
Critical Section Example 2

Code (for input size $n$)

1. A
2. for (int i = 0; i < n; i++)
3. B
4. for (int j = 0; j < n; j++)
5. C
6. D

Code execution

- A $\Rightarrow$ once
- B $\Rightarrow$ n times
- C $\Rightarrow$ $n^2$ times
- D $\Rightarrow$ once

Time $\Rightarrow$ $1 + n + n^2 + 1 = O(n^2)$
Critical Section Example 3

Code (for input size $n$)

1. A
2. for (int $i = 0; i < n; i++$)
3. for (int $j = i+1; j < n; j++$)
4. B

Code execution

- A $\Rightarrow$ once
- B $\Rightarrow$ $\frac{1}{2} n (n-1)$ times

Time $\Rightarrow 1 + \frac{1}{2} n^2 = O(n^2)$
Critical Section Example 4

Code (for input size $n$)

1. A
2. for (int $i = 0; i < n; i++$)
3. for (int $j = 0; j < 10000; j++$)
4. B

Code execution

- A ⇒ once
- B ⇒ 10000 $n$ times

Time ⇒ $1 + 10000 \cdot n = O(n)$
Critical Section Example 5

**Code (for input size $n$)**

1. `for (int i = 0; i < n; i++)`
2. `for (int j = 0; j < n; j++)`
3. `A`
4. `for (int i = 0; i < n; i++)`
5. `for (int j = 0; j < n; j++)`
6. `B`

**Code execution**

- $A \Rightarrow n^2$ times
- $B \Rightarrow n^2$ times

**Time** $\Rightarrow n^2 + n^2 = O(n^2)$
Critical Section Example 6

- Code (for input size $n$)
  1. $i = 1$
  2. while ($i < n$) {
  3.   A
  4.   $i = 2 \times i$
  5.   B
- Code execution
  - A $\Rightarrow \log(n)$ times
  - B $\Rightarrow 1$ times
- Time $\Rightarrow \log(n) + 1 = O(\log(n))$
Critical Section Example 7

Code (for input size n)

1. DoWork (int n)
2. if (n == 1)
3. A
4. else
5. DoWork(n/2)
6. DoWork(n/2)

Code execution

- A ⇒ 1 times
- DoWork(n/2) ⇒ 2 times

Time(1) ⇒ 1  Time(n) = 2 × Time(n/2) + 1
Recursive Algorithms

Definition

An algorithm that calls itself

Components of a recursive algorithm

1. Base cases
   - Computation with no recursion
2. Recursive cases
   - Recursive calls
   - Combining recursive results
Recursive Algorithm Example

Code (for input size n)

1. DoWork (int n)
2. if (n == 1)
3. A
4. else
5. DoWork(n/2)
6. DoWork(n/2)
Comparing Complexity

- Compare two algorithms
  - \( f(n), g(n) \)

- Determine which increases at faster rate
  - As problem size \( n \) increases

- Can compare ratio
  - If \( \infty \), \( f() \) is larger
  - If \( 0 \), \( g() \) is larger
  - If constant, then same complexity

\[
\lim_{{n \to \infty}} \frac{f(n)}{g(n)}
\]
Complexity Comparison Examples

- $\log(n)$ vs. $n^{\frac{1}{2}}$

  $$\lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \lim_{n \to \infty} \frac{\log(n)}{n^{\frac{1}{2}}} \quad \rightarrow \quad 0$$

- $1.001^n$ vs. $n^{1000}$

  $$\lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \lim_{n \to \infty} \frac{1.001^n}{n^{1000}} \quad \rightarrow \quad ??$$

  Not clear, use L’Hopital’s Rule
**Additional Complexity Measures**

- **Upper bound**
  - Big-O \(\Rightarrow O(\ldots)\)
  - Represents upper bound on # steps

- **Lower bound**
  - Big-Omega \(\Rightarrow \Omega(\ldots)\)
  - Represents lower bound on # steps

- **Combined bound**
  - Big-Theta \(\Rightarrow \Theta(\ldots)\)
  - Represents combined upper/lower bound on # steps
  - Best possible asymptotic solution
2D Matrix Multiplication Example

**Problem**

\( C = A \times B \)

**Lower bound**

\( \Omega(n^2) \)  

Required to examine 2D matrix

**Upper bounds**

\( O(n^3) \)  

Basic algorithm

\( O(n^{2.807}) \)  

Strassen’s algorithm (1969)

\( O(n^{2.376}) \)  

Coppersmith & Winograd (1987)

**Improvements still possible (open problem)**

Since upper & lower bounds do not match
### Additional Complexity Categories

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Deterministic polynomial time</td>
</tr>
<tr>
<td>NP</td>
<td>Nondeterministic polynomial time</td>
</tr>
<tr>
<td>PSPACE</td>
<td>Polynomial space</td>
</tr>
<tr>
<td>EXPSPACE</td>
<td>Exponential space</td>
</tr>
<tr>
<td>Decidable</td>
<td>Can be solved by finite algorithm</td>
</tr>
<tr>
<td>Undecidable</td>
<td>Not solvable by finite algorithm</td>
</tr>
</tbody>
</table>

If a problem has an algorithm that solves it in time $X$, then the problem is said to be in $X$

- e.g., matrix multiplication is in P
Why do we care?

If a problem can’t be solved in P time, then no algorithm can solve all cases exactly in a reasonable amount of time.

- But there might be an algorithm that always quickly gives close approximations.
- Or an algorithm that gives quick exact answers for the cases we care about.

Issues such as PSPACE vs. EXPSPACE are interesting theoretical issues, and sometimes provide practice insight.
NP Time Algorithms

Two ways of thinking about it

First way:
- Given a problem, and a potential answer, can we verify that the answer is correct in polynomial time?

Second way:
- Whenever the algorithm wants, it can clone itself in two
- If either clone finds an answer, the entire system produces an answer
Graph 3-coloring

- Given a graph (vertices and undirected edges)
- Can you find a way to color each vertex either blue, red, or green
- Such that no two vertices connected by an edge have the same color?
Some graphs are hard
3 coloring a graph is in NP

- There exist NP algorithms to 3 color a graph
  - Guess a 3 coloring
  - Verify that the coloring is valid
    - Easy to do in $O(n)$ time

- No one knows if there exists a polynomial time algorithm to find a 3 coloring for an arbitrary graph
  - If you come up with one, even one that runs in time $O(n^{100})$, you win one million dollars
  - Seriously
NP Time Algorithm

Many interesting problems are solvable with an NP algorithm, but not known to be solvable with a P algorithm

- Boolean satisfiability
- Traveling salesman problem (TLP)
- Bin packing

Key to solving many optimization problems

- Most efficient trip routes
- Most efficient schedule for employees
- Most efficient usage of resources
Some problems are NP-complete

They are in NP

And if a polynomial time algorithm existed for the program

Then every single last problem in NP could be solved in polynomial time

Almost all problems that are in NP but are not known to be in P are NP-complete

But not all
P = NP?

Are NP problems solvable in polynomial time?

- Prove $P=NP$
  - Show polynomial time solution exists for any NP-complete problem

- Prove $P \neq NP$
  - Show no polynomial-time solution possible for some problem in NP
  - The expected answer

The most important open problem in CS

- $1$ million prize offered by Clay Math Institute
- Plus front page, NY Times, job offers galore, instant Ph.D. in Computer Science
Algorithmic Complexity Summary

- Asymptotic complexity
  - Fundamental measure of efficiency
  - Independent of implementation & computer platform

- Learned how to
  - Examine program
  - Find critical sections
  - Calculate complexity of algorithm
  - Compare complexity