CMSC330 Fall 2009 Example Quiz #3 Solutions

1. (9 pts) Lambda calculus

   (3 pts) Find all free (unbound) variables in the following λ-expression

   a. (λa. c b) λb. a

      (λa. c b) λb. a   **// rightmost a**
      // b in body of 1^{st} λ
      // c

   (3 pts each) Evaluate the following λ-expressions as much as possible

   b. (λx.λy.y x) a b

      (λx.λy.y x) a b → (λy.y a) b → b a

   c. (λz.z x) (λy.y x)

      (λz.z x) (λy.y x) → (λy.y x) x → x x

2. (16 pts) Lambda calculus encodings

   Prove the following using the appropriate λ-calculus encodings, given**:**

   1 = λf.λy.f y
   2 = λf.λy.f (f y)
   3 = λf.λy.f (f (f y))
   4 = λf.λy.f (f (f (f y)))
   M * N = λx.(M (N x))
   Y = λf.(λx.f (x x)) (λx.f (x x))
   succ = λz.λf.λy.f (z f y)

   a. (10 pts) 2 * 2 =>* 4

(2 * 2)	**// replacing * w/ encoding**
= λx.(2 (2 x))	**// (1 pt) replacing 2 w/ encoding**
= λx.(2 ((λf.λy.f (f y)) x))	**// (1 pts) β-reduction: f → x**
= λx.(2 (λy.x (x y)))	**// (1 pt) replacing 2 w/ encoding**
= λx.((λf.λy.f (f y)) (λy.x (x y)))	**// (2 pts) a-conversion: y → a**
= λx.((λf.λa.f (f a)) (λy.x (x y)))	**// (1 pts) β-reduction: f → λy.x (x y)**
= λx.(λa. (λy.x (x y)) ((λy.x (x y))a))	**// (1 pts) β-reduction: 2^{nd} y → a**
= λx.(λa. (λy.x (x y)) (x (x a)))	**// (1 pts) β-reduction: y → x (x a)**
= λx.(λa. x (x (x (x a))))	**// (1 pt) apply encoding for 4**
= 4	**// (1 pt) result**

   b. (6 pts) (Y succ) x =>* succ (Y succ) x
      // you do not need to expand succ
      // you may assume ((λx.succ (x x)) (λx.succ (x x))) => (Y succ)

(Y succ) x	**// replace Y w/ encoding**
= (λf.(λx.f (x x)) (λx.f (x x)) succ) x	**// (2 pts) 1^{st} f → succ**
= (λx.succ (x x)) (λx.succ (x x)) x	**// (2 pts) 1^{st} x → λx.succ (x x)**
= (succ ((λx.succ (x x)) (λx.succ (x x)))) x	**// (1 pts) encoding for (Y succ)**
= (succ (Y succ)) x	**// (1 pts) result**