CMSC 451: Interval Scheduling

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Based on Section 4.1 of *Algorithm Design* by Kleinberg & Tardos.
Interval Scheduling

- You want to schedule jobs on a supercomputer.
- Requests take the form \((s_i, f_i)\) meaning a job that runs from time \(s_i\) to time \(f_i\).
- You get many such requests, and you want to process as many as possible, but the computer can only work on one job at a time.

\[
\text{Interval Scheduling}
\]

Given a set \(J = \{(s_i, f_i) : i = 1, \ldots, n\}\) of job intervals, find the largest \(S \subset J\) such that no two intervals in \(S\) overlap.
Greedy Algorithm

Greedy Algorithms

- Not easy to define what we mean by “greedy algorithm”
- Generally means we take little steps, looking only at our local choices
- Often among the first reasonable algorithms we can think of
- Frequently doesn’t lead to optimal solutions, but sometimes it does.
Example Greedy Algorithms

- TreeGrowing is an example of a greedy framework for most choices of `nextEdge` functions.

- Topological sort & testing bipartiteness were greedy algorithms

- Interval Scheduling turns out to have a nice greedy algorithm that works.
Ideas for Interval Scheduling

A greedy framework:

\[ S = \text{set of input intervals } (s_i, f_i) \]

While S is not empty:

1. \( q = \text{nextInterval}(S) \)
2. Output interval \( q \)
3. Remove intervals that overlap with \( q \) from \( S \)

What are possible rules for nextInterval?
What are possible rules for `nextInterval`?

1. *Choose the interval that starts earliest.*
   Rationale: start using the resource as soon as possible.

2. *Choose the smallest interval.*
   Rationale: try to fit in lots of small jobs.

3. *Choose the interval that overlaps with the fewest remaining intervals.*
   Rationale: keep our options open and eliminate as few intervals as possible.
Rules That Don’t Work

1. Earliest start time

2. Shortest job

3. Fewest conflicts
Rules That Don’t Work

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Rules That Don’t Work

1. Earliest start time

2. Shortest job

3. Fewest conflicts
Optimal Greedy Algorithm

- Choose the interval with the earliest finishing time.
  Rationale: ensure we have as much of the resource left as possible.

\[
S = \text{set of input intervals } \{(s[i], f[i])\}
\]
While S is not empty:
  \(q = \text{a request in } S \text{ that has the}
  \text{soonest finishing time}\)
  Output interval q
  Remove intervals that overlap with q from S

- This algorithm chooses a compatible set of intervals.
How to Prove Optimality

How can we prove the schedule returned is optimal?

• Let $A$ be the schedule returned by this algorithm.
• Let $OPT$ be some optimal solution.

Might be hard to show that $A = OPT$, instead we need only to show that $|A| = |OPT|$.

Note the distinction: instead of proving directly that a choice of intervals $A$ is the same as an optimal choice, we prove that it has the same number of intervals as an optimal. Therefore, it is optimal.
Let these be the schedules of $A$ and $OPT$:

$A$: $i_1, i_2, \ldots, i_k$

$OPT$: $j_1, j_2, \ldots, j_m$

Let $f(i)$ be the finishing time of job $i$.

**Theorem**

For all $r \leq k$ we have $f(i_r) \leq f(j_r)$.

**Proof.**

By induction. True when $r = 1$ because we chose greedily.

Assume $f(i_{r-1}) \leq f(j_{r-1})$. Then we have

$$f(i_{r-1}) \leq f(j_{r-1}) \leq s(j_r),$$

where $s(j_r)$ is the start time of job $j_r$. So, job $j_r$ is available when the greedy algorithm makes its choice. Hence, $f(i_r) \leq f(j_r)$. □
Theorem

This greedy algorithm is optimal for Interval Scheduling.

Proof.

Suppose not. Then if $A$ has $k$ jobs, OPT has $m > k$ jobs.

By our lemma, after $k$ jobs we have this situation:

So, job $j_{k+1}$ must have been available to the greedy algorithm.
Implementation

1. Sort the intervals based on $f_i$ — takes $O(n \log n)$.

2. Scan down this list, output the first element that starts after the finishing time of the last item output — takes $O(n)$.

\[
\begin{array}{cccccccc}
    f_1 & f_2 & f_3 & f_4 & f_5 & f_5 & f_6 & f_7 \\
    s_1 & s_2 & s_3 & s_4 & s_5 & s_5 & s_6 & s_7 \\
\end{array}
\]

LatestFinishingTime = finishing time of last scheduled interval
Extensions

• Online algorithms: What if you don’t know all the intervals at the start? Current active area of research.

• What if some intervals are more important than others? Weighted interval scheduling. — We’ll see this later.
Other rules of scheduling intervals also lead to nice greedy algorithms. For example:
Interval Partitioning Problem

Given intervals \((s_i, f_i)\) assign them to processors so that you schedule every interval and use the smallest \# of processors.

Now, we’re trying to minimize the \# of processors (or rooms) used.

Another way to think of it: Each processor corresponds to a color. We’re trying to color intervals with the fewest \# of colors so that no two overlapping intervals get the same color.
Interval Partitioning Problem

**Definition**

Depth is the maximum number of intervals passing over any time point.

**Theorem**

The number of processors needed is at least the depth of the set of intervals.

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depth = 3
```
We need at least depth processors.

Can we find a schedule with no more than depth processors?
Greedy Alg for Interval Partitioning

Yes: Let \( \{1, \ldots, d\} \) be a set of labels, where \( d = \text{depth} \).

Sort intervals by start time
For \( j = 1, 2, 3, \ldots, n \)
Let \( Q = \) set of labels that haven’t been assigned to a preceding interval that overlaps \( I_j \)
If \( Q \) is not empty,
Pick any label from \( Q \) and assign it to \( I_j \)
Else
Leave \( I_j \) unlabeled
Endfor
Every interval gets a label

No overlapping intervals get the same label because we exclude the labels that have already been used.

Every interval gets a label: The only way it wouldn’t is if we’ve run out of labels. That would mean, when coloring interval $I$ that $\geq d$ intervals with start times before $I$ overlap $I$:

So, when coloring $i$, there must be a color free.
Given a graph $G$ and a number $k$, color the nodes with $k$ colors such that no edge connects 2 vertices of the same color (or report that it can’t be done).
Graph Coloring

Given a graph $G$ and a number $k$, color the nodes with $k$ colors such that no edge connects 2 vertices of the same color (or report that it can’t be done).
Interval Graph Coloring

When \( k \geq 3 \), Graph Coloring is hard in general.

We saw an algorithm for Graph Coloring when \( k = 2 \).

**Interval Graph Coloring**

Given a graph \( G \) derived from a set of intervals and a number \( k \), color the nodes with \( k \) colors such that no edge connects 2 vertices of the same color (or report that it can’t be done).

Now we’ve seen an algorithm for solving Graph Coloring on the restricted class of graphs called “Interval Graphs.”