Information CMSC251

Asymptotic Notations.

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.

$\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.

$f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$f(n) = \omega(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$f(n) \sim g(n)$ if $f(n) = g(n) + o(g(n))$.

Logarithms.

$$a = b^{\log_b a} \qquad \log_c(ab) = \log_c a + \log_c b \qquad \log_b a^n = n \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b(1/a) = -\log_b a \qquad \log_b a = \frac{1}{\log_a b} \qquad a^{\log_b n} = n^{\log_b a}$$

Stirling's Formula.

$$n! \;\approx\; \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Recurrences "Master Theorem":

$$T(n) = \begin{cases} aT(n/b) + cn^d & n > 1 \\ f & n = 1 \end{cases}$$

implies

$$T(n) = \begin{cases} \left(f + \frac{c}{ab^{-d}-1}\right) n^{\log_b a} - \frac{cn^d}{ab^{-d}-1} = \begin{cases} \Theta(n^{\log_b a}) & a > b^d \\ \Theta(n^d) & a < b^d \end{cases} \\ n^d(f + c\log_b n) = \Theta(n^d \log_b n) & a = b^d \end{cases}.$$

`Summing solutions:` If

$$T(n) = \begin{cases} aT(n/b) + \sum c_i n^{d_i} & n > 1 \\ f & n = 1 \end{cases}$$

then we can just sum the solutions of each recurrence:

$$T_i(n) = \begin{cases} aT_i(n/b) + c_i n^{d_i} & n > 1 \\ 0 & n = 1 \end{cases}$$

and add in $fn^{\log_b a}$ for the contribution from the leaves.

Summations.

Distribution law:

$$\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_i b_j\right)$$

Interchanging order of summation:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}$$

Splitting range:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{r} a_k + \sum_{k=r+1}^{n} a_k$$

Arithmetic series:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometric series:

$$\sum_{k=0}^{n} x^k = 1 + x + +x^2 \cdots + x^n = \frac{x^{n+1}-1}{x-1} \qquad x \neq 1$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad x < 1$$

Harmonic series:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$

Telescoping series:

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$$

Products:

$$\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n \qquad\qquad \log \prod_{k=1}^{n} a_k = \sum_{k=1}^{n} \log a_k$$

Approximation by integrals:

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx \qquad \text{for } f(x) \text{ monotonically increasing}$$

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx \qquad \text{for } f(x) \text{ monotonically decreasing}$$

Quadratic Formula.

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Probability.

$$\mathrm{E}[X] = \sum_x x \Pr\{X = x\}\ , \qquad \mathrm{Var}[X] = \mathrm{E}[(X - \mathrm{E}(X))^2] = \mathrm{E}[X^2] - \mathrm{E}^2[X]\ , \qquad \sigma[X] = \sqrt{\mathrm{Var}[X]}\ .$$