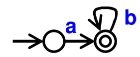
# CMSC 330: Organization of Programming Languages

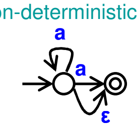
Finite Automata 2

## Last Lecture

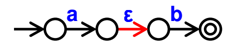
- Finite automata
  - Alphabet, states…
  - ($\Sigma$, Q, $q_0$, F, $\delta$)
- Types
  - Deterministic (DFA)



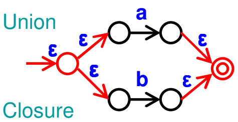
  - Non-deterministic (NFA)



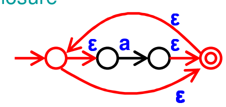
- Reducing RE to NFA
  - Concatenation



  - Union



  - Closure





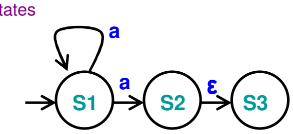
## This Lecture

- Reducing NFA to DFA
  - ε-closure
  - Subset algorithm
- Minimizing DFA
  - Hopcroft reduction
- Complementing DFA
- Implementing DFA



## How NFA Works

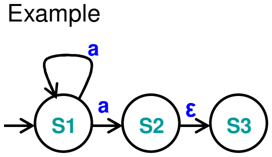
- When NFA processes a string
  - NFA may be in several possible states
    - Multiple transitions with same label
    - ε-transitions
- Example
  - After processing "a"
    - NFA may be in states
      S1
      S2
      S3



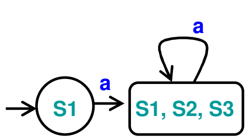



## Reducing NFA to DFA

- NFA may be reduced to DFA
  - By explicitly tracking the set of NFA states
- Intuition
  - Build DFA where
    - Each DFA state represents a set of NFA states
- Example




NFA




DFA



## Reducing NFA to DFA (cont.)

- Reduction applied using the subset algorithm
  - DFA state is a subset of set of all NFA states
- Algorithm
  - Input
    - NFA ($\Sigma$, Q, $q_0$, $F_n$, $\delta$)
  - Output
    - DFA ($\Sigma$, R, $r_0$, $F_d$, $\delta$)
  - Using
    - ε-closure(p)
    - move(p, a)

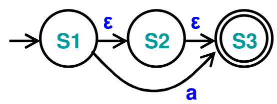
## ε-transitions and ε-closure

- We say $p \xrightarrow{\varepsilon} q$
  - If it is possible to go from state p to state q by taking only ε-transitions
  - If $\exists$ p, $p_1$, $p_2$, … $p_n$, q $\in$ Q such that
    - {p,ε,$p_1$} $\in \delta$, {$p_1$,ε,$p_2$} $\in \delta$, … , {$p_n$,ε,q} $\in \delta$
- ε-closure(p)
  - Set of states reachable from p using ε-transitions alone
    - Set of states q such that $p \xrightarrow{\varepsilon} q$
    - ε-closure(p) = {q | $p \xrightarrow{\varepsilon} q$ }
  - Note
    - ε-closure(p) always includes p
    - ε-closure( ) may be applied to set of states (take union)



## ε-closure: Example 1

- Following NFA contains
  - S1 $\xrightarrow{\varepsilon}$ S2
  - S2 $\xrightarrow{\varepsilon}$ S3
  - S1 $\xrightarrow{\varepsilon}$ S3

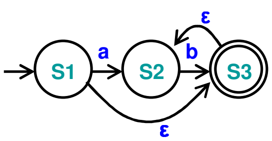



- ε-closures
  - ε-closure(S1) = { S1, S2, S3 }
  - ε-closure(S2) = { S2, S3 }
  - ε-closure(S3) = { S3 }
  - ε-closure( { S1, S2 } ) = { S1, S2, S3 } $\cup$ { S2, S3 }



## ε-closure: Example 2

- Following NFA contains
  - S1 $\xrightarrow{\varepsilon}$ S3
  - S3 $\xrightarrow{\varepsilon}$ S2
  - S1 $\xrightarrow{\varepsilon}$ S2

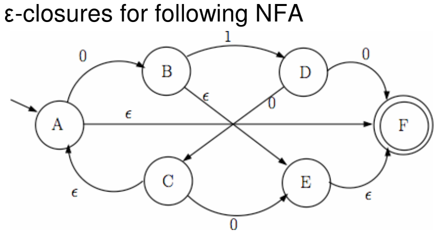



- ε-closures
  - ε-closure(S1) = { S1, S2, S3 }
  - ε-closure(S2) = { S2 }
  - ε-closure(S3) = { S2, S3 }
  - ε-closure( { S2,S3 } ) = { S2 } $\cup$ { S2, S3 }



## ε-closure: Practice

- Find ε-closures for following NFA




- Find ε-closures for the NFA you construct for
  - The regular expression (0|1*)111(0*|1)



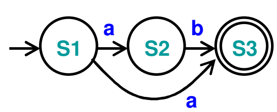
## Calculating move(p,a)

- move(p,a)
  - Set of states reachable from p using exactly one transition on a
    - Set of states q such that {p, a, q} $\in \delta$
    - move(p,a) = {q | {p, a, q} $\in \delta$}
  - Note move(p,a) may be empty Ø
    - If no transition from p with label a



## move(a,p) : Example 1

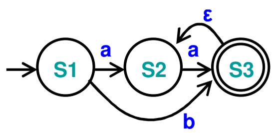
- Following NFA
  - Σ = { a, b }




- Move
  - move(S1, a) = { S2, S3 }
  - move(S1, b) = Ø
  - move(S2, a) = Ø
  - move(S2, b) = { S3 }
  - move(S3, a) = Ø
  - move(S3, b) = Ø

## move(a,p) : Example 2

- Following NFA
  - Σ = { a, b }




- Move
  - move(S1, a) = { S2 }
  - move(S1, b) = { S3 }
  - move(S2, a) = { S3 }
  - move(S2, b) = Ø
  - move(S3, a) = Ø
  - move(S3, b) = Ø



## NFA → DFA Reduction Algorithm

- Input NFA $(\Sigma, Q, q_0, F_n, \delta)$, Output DFA $(\Sigma, R, r_0, F_d, \delta)$
- Algorithm

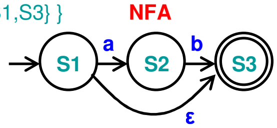
| | |
|---|---|
| Let $r_0$ = ε-closure($q_0$), add it to R | // DFA start state |
| While ∃ an unmarked state r ∈ R | // process DFA state r |
| Mark r | // each state visited once |
| For each a ∈ Σ | // for each letter a |
| Let S = {s \| q ∈ r & move(q,a) = s} | // states reached via a |
| Let e = ε-closure(S) | // states reached via ε |
| If e ∉ R | // if state e is new |
| Let R = e ∪ R | // add e to R (unmarked) |
| Let δ = δ ∪ {r, a, e} | // add transition r→e |
| Let $F_d$ = {r \| ∃ s ∈ r with s ∈ $F_n$} | // final if include state in $F_n$ |



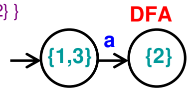
## NFA → DFA Example 1

- Start = ε-closure(S1) = { {S1,S3} }
- R = { {S1,S3} }
- r ∈ R = {S1,S3}
- Move({S1,S3},a) = {S2}
  - e = ε-closure({S2}) = {S2}
  - R = R ∪ {S2} = { {S1,S3}, {S2} }
  - δ = δ ∪ {{S1,S3}, a, {S2}}
- Move({S1,S3},b) = Ø

**NFA**




**DFA**

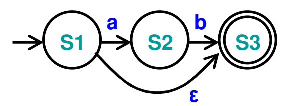




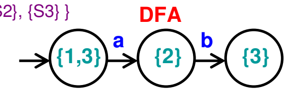

## NFA → DFA Example 1 (cont.)

- R = { {S1,S3}, {S2} }
- r ∈ R = {S2}
- Move({S2},a) = Ø
- Move({S2},b) = {S3}
  - e = ε-closure({S3}) = {S3}
  - R = R ∪ {S3} = { {S1,S3}, {S2}, {S3} }
  - δ = δ ∪ {{S2}, b, {S3}}

**NFA**




**DFA**

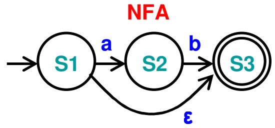




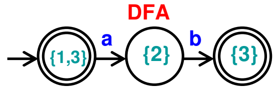

## NFA → DFA Example 1 (cont.)

- R = { {S1,S3}, {S2}, {S3} }
- r ∈ R = {S3}
- Move({S3},a) = Ø
- Move({S3},b) = Ø
- $F_d$ = {{S1,S3}, {S3}}
  - Since S3 ∈ $F_n$
- Done!

**NFA**



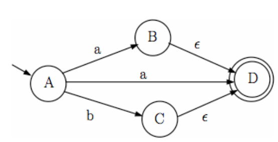

**DFA**



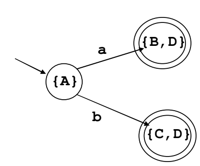



## NFA → DFA Example 2

- NFA



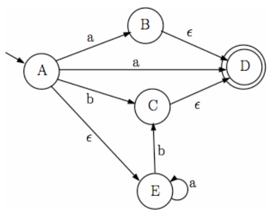

- DFA



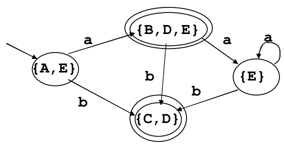

R = { {A}, {B,D}, {C,D} }

## NFA → DFA Example 3

- NFA




- DFA

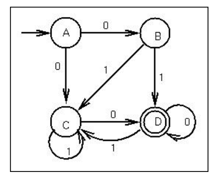



R = { {A,E}, {B,D,E}, {C,D}, {E} }

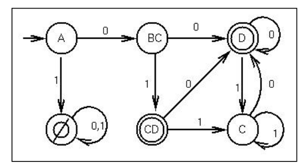


## Equivalence of DFAs and NFAs

- Any string from {A} to either {D} or {CD}
  - Represents a path from A to D in the original NFA




NFA




DFA



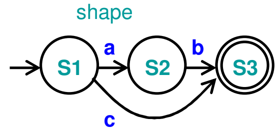
## Equivalence of DFAs and NFAs (cont.)

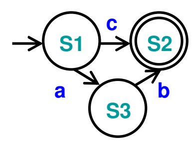
- Can reduce any NFA to a DFA using subset alg.
- How many states in the DFA?
  - Each DFA state is a subset of the set of NFA states
  - Given NFA with n states, DFA may have $2^n$ states
    - Since a set with n items may have $2^n$ subsets
  - Corollary
    - Reducing a NFA with n states may be $O(2^n)$



## Minimizing DFA

- Result from CS theory
  - Every regular language is recognizable by a minimum-state DFA that is unique up to state names
- In other words
  - For every DFA, there is a unique DFA with minimum number of states that accepts the same language
  - Two minimum-state DFAs have same underlying shape






## Minimizing DFA: Hopcroft Reduction

- Intuition
  - Look for states that can be distinguish from each other
    - End up in different accept / non-accept state with identical input
- Algorithm
  - Construct initial partition
    - Accepting & non-accepting states
  - Iteratively refine partitions (until partitions remain fixed)
    - Split a partition if members in partition have transitions to different partitions for same input
      - Two states x, y belong in same partition if and only if for all symbols in Σ they transition to the same partition
  - Update transitions & remove dead states

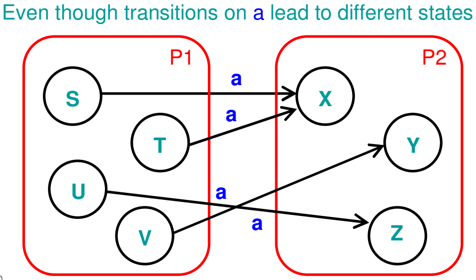


J. Hopcroft, "An n log n algorithm for minimizing states in a finite automaton," 1971



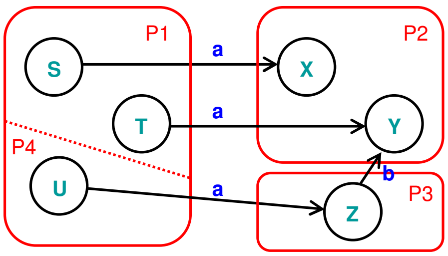
## Splitting Partitions

- No need to split partition {S,T,U,V}
  - All transitions on a lead to identical partition P2
  - Even though transitions on a lead to different states



## Splitting Partitions (cont.)

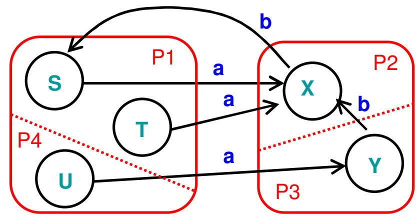
- Need to split partition {S,T,U} into {S,T}, {U}
  - Transitions on a from S,T lead to partition P2
  - Transition on a from R lead to partition P3






## Resplitting Partitions

- Need to reexamine partitions after splits
  - Initially no need to split partition {S,T,U}
  - After splitting partition {X,Y} into {X}, {Y}
  - Need to split partition {S,T,U} into {S,T}, {U}






## DFA Minimization Algorithm (1)

- Input DFA ($\Sigma$, Q, $q_0$, $F_n$, $\delta$), Output DFA ($\Sigma$, R, $r_0$, $F_d$, $\delta$)
- Algorithm

```
Let p0 = Fn, p1 = Q – F          // initial partitions = final, nonfinal states
Let R = { p | p ∈ {p0,p1} and p != Ø }, P = Ø // add p to R if nonempty
While P != R do                  // while partitions changed on prev iteration
    Let P = R, R = Ø             // P = prev partitions, R = current partitions
    For each p ∈ P               // for each partition from previous iteration
        {p0,p1} = split(p,P)     // split partition, if necessary
        R = R ∪ { p | p ∈ {p0,p1} and p != Ø } // add p to R if nonempty
r0 = p ∈ R where q0 ∈ p          // partition w/ starting state
Fd = { p | p ∈ R and exists s ∈ p such that s ∈ Fn } // parts. w/ final states
δ(p,c) = q when δ(s,c) = r where s ∈ p and r ∈ q    // add transitions
```



## DFA Minimization Algorithm (2)

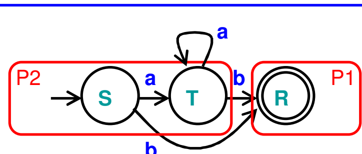
- Algorithm for split(p,P)

```
Choose some r ∈ p, let q = p – {r}, m = { } // pick some state r in p
For each s ∈ q                    // for each state in p except for r
    For each c ∈ Σ                // for each symbol in alphabet
        If δ(r,c) = q0 and δ(s,c) = q1 and // q's = states reached for c
            there is no p1 ∈ P such that q0 ∈ p1 and q1 ∈ p1 then
            m = m ∪ {s}           // add s to m if q's not in same partition
Return p – m, m                   // m = states that behave differently than r
                                  // m may be Ø if all states behave the same
                                  // p – m = states that behave the same as r
```

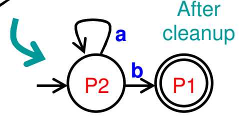


## Minimizing DFA: Example 1

- DFA




- Initial partitions
  - Accept { R } → P1
  - Reject { S, T } → P2

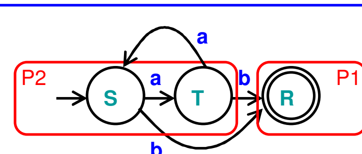



- Split partition? → Not required, minimization done
  - move(S,a) = T → P2 – move(S,b) = R → P1
  - move(T,a) = T → P2 – move (T,b) = R → P1

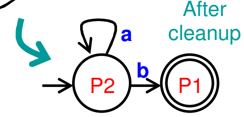


## Minimizing DFA: Example 2

- DFA



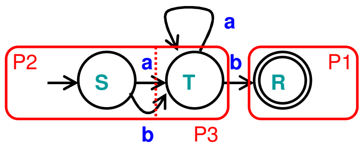

- Initial partitions
  - Accept { R } → P1
  - Reject { S, T } → P2




- Split partition? → Not required, minimization done
  - move(S,a) = T → P2 – move(S,b) = R → P1
  - move(T,a) = S → P2 – move (T,b) = R → P1

## Minimizing DFA: Example 3

- DFA



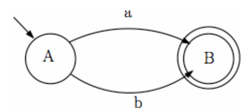

DFA already minimal

- Initial partitions
  - Accept { R } → P1
  - Reject { S, T } → P2
- Split partition? → Yes, different partitions for B
  - move(S,a) = T → P2 – move(S,b) = T → P2
  - move(T,a) = T → P2 – move (T,b) = R → P1



## Complement of DFA

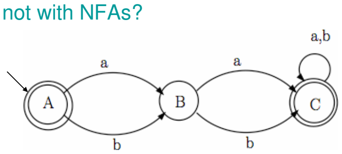
- Given a DFA accepting language L
  - How can we create a DFA accepting its complement?
  - Example DFA
    - Σ = {a,b}






## Complement of DFA (cont.)

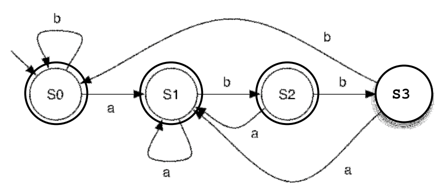
- Algorithm
  - Add explicit transitions to a dead state
  - Change every accepting state to a non-accepting state & every non-accepting state to an accepting state
- Note this only works with DFAs
  - Why not with NFAs?






## Practice

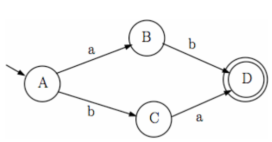
Make the DFA which accepts the complement of the language accepted by the DFA below.

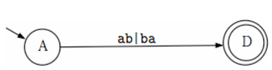





## Reducing DFAs to REs

- General idea
  - Remove states one by one, labeling transitions with regular expressions
  - When two states are left (start and final), the transition label is the regular expression for the DFA



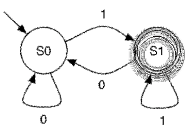



## Relating REs to DFAs and NFAs

- Why do we want to convert between these?
  - Can make it easier to express ideas
  - Can be easier to implement

## Implementing DFAs

It's easy to build a program which mimics a DFA




```
cur_state = 0;
while (1) {

  symbol = getchar();

  switch (cur_state) {

    case 0: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("rejected\n"); return 0;
              default:   printf("rejected\n"); return 0;
            }
            break;

    case 1: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("accepted\n"); return 1;
              default:   printf("rejected\n"); return 0;
            }
            break;

    default: printf("unknown state; I'm confused\n");
             break;
  }
}
```



## Implementing DFAs (Alternative)

Alternatively, use generic table-driven DFA

given components ($\Sigma$, Q, $q_0$, F, $\delta$) of a DFA:
let q = $q_0$
while (there exists another symbol s of the input string)
  q := $\delta$(q, s);
if q$\in$F then
  accept
else reject

- q is just an integer
- Represent $\delta$ using arrays or hash tables
- Represent F as a set



## Run Time of DFA

- How long for DFA to decide to accept/reject string s?
  - Assume we can compute $\delta$(q, c) in constant time
  - Then time to process s is O(|s|)
    - Can't get much faster!
- Constructing DFA for RE A may take $O(2^{|A|})$ time
  - But usually not the case in practice
- So there's the initial overhead
  - But then processing strings is fast



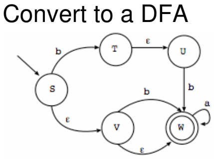
## Regular Expressions in Practice

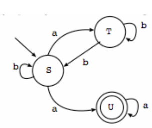
- Regular expressions are typically "compiled" into tables for the generic algorithm
  - Can think of this as a simple byte code interpreter
  - But really just a representation of ($\Sigma$, $Q_A$, $q_A$, $\{f_A\}$, $\delta_A$), the components of the DFA produced from the RE
- Regular expression implementations often have extra constructs that are non-regular
  - I.e., can accept more than the regular languages
  - Can be useful in certain cases
  - Disadvantages
    - Nonstandard, plus can have higher complexity



## Practice

- Convert to a DFA




- Convert to an NFA and then to a DFA
  - (0|1)*11|0*
  - Strings of alternating 0 and 1
  - aba*|(ba|b)



## Summary of Regular Expression Theory

- Finite automata
  - DFA, NFA
- Equivalence of RE, NFA, DFA
  - RE → NFA
    - Concatenation, union, closure
  - NFA → DFA
    - ε-closure & subset algorithm
- DFA
  - Minimization, complement
  - Implementation