LECTURE 3: SATISFIABILITY
SATISFIABILITY

Definition:
A compound proposition is **satisfiable** if it can be made true by assigning appropriate values to its **variables**.

Examples: Are these satisfiable?

\[
p \lor \neg p
\]

\[
p \land \neg p
\]

\[
(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)
\]

\[
(p \land q \land r) \land (p \land r \rightarrow \neg q)
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p \land \neg p \\
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\[ p \land \neg p \quad \text{No (contradiction)} \]
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Examples: Are these satisfiable?

\[
\begin{align*}
p \lor \lnot p & \quad \text{Yes (tautology)} \\
p \land \lnot p & \quad \text{No (contradiction)} \\
(p \lor \lnot q) \land (q \lor \lnot r) \land (r \lor \lnot p) \quad \text{Yes} \\
(p \land q \land r) \land (p \land r \rightarrow \lnot q) & 
\end{align*}
\]
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(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) & \quad \text{Yes} \\
(p \land q \land r) \land (p \land r \rightarrow \neg q) & \quad \text{No}
\end{align*}
\]
SAT SOLVERS

Definition:
The SAT problem is the problem of determining whether a compound proposition is satisfiable. If so, truth assignments that satisfy the compound proposition are called the solution.

SAT problems are HARD!
• NP Complete
• No efficient algorithm for ALL problems
• Truth table: 1000 variables = \( 2^{1000} > 10^{300} \) rows
• SAT solvers exists that can solve problems with many unknowns
REDUCTIONS TO SAT

Many problems can be mapped into a SAT problem
• Fact: Any “decidability” problem reduces to SAT
• SAT solvers can solve almost anything

Real-World:
• Register allocation
• Traveling salesman (delivery routing)
• Packing problems (knapsack)
• Cryptography
• Verifying digital circuits
• FPGS layout
• Candy Crush Saga

Examples we’ll look at:
• System Specification
• Logic puzzles
• Sudoku
SYSTEM SPECIFICATION

Logical requirements that a system must satisfy:
• Written before coding begins
• Unit tests build to verify system specifications

“If a message is stored in buffer, it is transmitted”
“A message is stored in the buffer”

Definition:
Specifications are **consistent** if they can be satisfied simultaneously. Otherwise, they are **inconsistent**.

“The message is not transmitted”
LOGIC PUZZLES

His Fault

- Word Puzzles that reduce to SAT problems
- Big part of LSAT

Knights: Always tell the truth
Knaves: Compulsive liars

Raymond Smullyan
FAMOUS PUZZLE:

You go to an island and meet A and B.
A says: “B is a knight”
B says: “The two of us are of opposite types”

Convert to SAT:
p: A is a knight
q: B is a knight

Do on Board
APPLICATION: THE LABYRINTH

*Hypothetical Situation*

You’re Jennifer Connelly and you’re in the 1986 film “The Labyrinth” with David Bowie. You’re walking through a maze trying to find your little brother that David Bowie has kidnapped, and for some reason David Bowie is wearing leggings and a dance belt. You come to a fork in the road with two strange creatures. They tell you one path leads to death and one leads to freedom. One creature is a knight and the other is a knave.

You ask A: “what would B say if I asked him which way to freedom?” A responds “left.” Which way should you go?
Rules:
- Every column has every digit 1-9
- Every row has every digit 1-9
- Every block has every digit 1-9
- Every square has 1 digit

Define simple propositions

\( p_{r,c,n} \) The box at row \( r \) and column \( c \) contains \( n \)

There are \( 9 \times 9 \times 9 = 729 \) variables

Idea: Embed into SAT
Cook up a compound proposition that is true only for solutions
Every row has every digit:
\[ \bigwedge \bigwedge \bigvee p_{r,c,n} \]
\[ r=1 \quad n=1 \quad c=1 \]

Every col has every digit:
\[ \bigwedge \bigwedge \bigvee p_{r,c,n} \]
\[ c=1 \quad n=1 \quad r=1 \]

Every block has every digit:
\[ \bigwedge \bigwedge \bigwedge \bigvee p_{3R+r,3C+c,n} \]
\[ R=0 \quad C=0 \quad n=0 \quad r=1 \quad c=1 \]
LOGIC RULES

Every cell has at most one number:

\[ \bigwedge_{r,c,n \neq n'} p_{r,c,n} \rightarrow \neg p_{r,c,n'} \]

Every “given” cell is already fixed:

\[ p_{1,1,5} = T \]
\[ p_{1,2,3} = T \]
\[ \vdots \]

Q: Do we need to assert that every cell has a number?