Name & UID: 

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CMSC250 Homework 9       Due: Wednesday, December 3, 2014
1. (15 points) Consider the following sequence:

\[
\begin{align*}
    a_0 &= 1 \\
    a_1 &= 1 + 2a_0 = 3 \\
    a_2 &= 1 + 2a_0 + 2a_1 = 9 \\
    a_3 &= 1 + 2a_0 + 2a_1 + 2a_2 = 27 \\
    \vdots \\
    a_n &= 1 + \sum_{i=0}^{n-1} 2a_i
\end{align*}
\]

We will use **Strong Induction** to prove that \( a_n = 3^n \) for all \( n \geq 0 \).

(a) What is the Base Case?

**Solution:**
\[
\begin{align*}
    n &= 0 \\
    \text{Left side} &= 1 \\
    \text{Right side} &= 3^0 = 1
\end{align*}
\]

(b) What is the Inductive Hypothesis?

**Solution:** Assume that for all \( 0 \leq k < n \), \( a_k = 3^k \).

(c) What is the Inductive Step?

**Solution:**
\[
\begin{align*}
    a_n &= 1 + \sum_{i=0}^{n-1} 2a_i \quad \text{by definition} \\
    &= 1 + 2\sum_{i=0}^{n-1} a_i \quad \text{by algebra} \\
    &= 1 + 2\sum_{i=0}^{n-1} 3^i \quad \text{by IH} \\
    &= 1 + 2\frac{3^n - 1}{3 - 1} \quad \text{sum of geometric series} \\
    &= 3^n \quad \text{by algebra}
\end{align*}
\]
2. (15 points) We will use Strong Induction to show that every positive integer \( n \) can be written as the sum of distinct powers of 2 (i.e., the integers \( 2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8 \), etc.).

(a) What is the Base Case.

**Solution:** For \( n = 1 \): \( 1 = 2^0 \).

(b) What is the Inductive Hypothesis.

**Solution:** Assume that for all \( 1 \leq m < n \), \( m \) can be written as the sum of distinct powers of 2.

(c) What is the Inductive Step. [Hint: Consider the cases of when \( n \) is even and odd separately.]

**Solution:** If \( n \) is even then by the IH \( n/2 \) can be written as the sum of distinct powers of 2. Multiplying each of these powers by 2 gives \( n \) as the sum of distinct powers of 2. If \( n \) is odd then by the IH \( (n - 1)/2 \) can be written as the sum of distinct powers of 2. Multiplying each of these powers by 2 and adding 1 = \( n^0 \) gives \( n \) as the sum of distinct powers of 2.
3. (15 points) Show that every positive integer $n$ can be written as the sum of distinct powers of 2 in a unique way. [Hint: Do not use Mathematical Induction.]

**Solution:** Proof by Contradiction. Assume $n$ can be written as the sum of distinct powers of 2 in two different ways. Find the smallest powers of 2 where the two ways disagree, say $2^a$ and $2^b$. Assume, WLOG than $a < b$. Subtract all of powers of 2 that are smaller than $2^a$ from both ways, producing a (possibly smaller) number that is the sum of distinct powers of 2 in two different ways. Divide both numbers by $2^a$ producing a (possibly smaller) number that is the sum of distinct powers of 2 in two different ways. The way that had $2^a$ as a power now has $2^0 = 1$ as a power and is therefore odd, but the other way is even. Thus they cannot be equal, which is a contradiction.
4. (15 points) Assume that you guess that a formula for
\[ \sum_{k=1}^{n} k2^k \]
has the form \( an2^n + b2^n + c \), where \( n \geq 1 \). We will use Constructive Mathematical Induction to derive a formula for the sum.

(a) What do you learn from the Base Case.

**Solution:** For \( n = 1 \)
\[
\sum_{k=1}^{n} k2^k = \sum_{k=1}^{1} k2^k = 1 \cdot 2^1 = 2 .
\]
\[ an2^n + b2^n + c = a \cdot 1 \cdot 2^1 + b \cdot 2^1 + c = 2a + 2b + c . \]
So
\[ 2a + 2b + c = 2 . \]

(b) What is the Inductive Hypothesis?

**Solution:** Assume for \( n - 1 \)
\[
\sum_{k=1}^{n-1} k2^k = a(n-1)2^{n-1} + b2^{n-1} + c .
\]

(c) Show the Inductive Step.

**Solution:**
\[
\sum_{k=1}^{n} k2^k = \sum_{k=1}^{n-1} k2^k + n2^n
\]
= \[ a(n-1)2^{n-1} + b2^{n-1} + c + n2^n \text{ by the IH} \]
= \[ \frac{a}{2}(n-1)2^n + \frac{b}{2}2^n + c + n2^n \]
= \[ \left( \frac{a}{2} + 1 \right)n2^n + \left( \frac{b-a}{2} \right)2^n + c \]
= \[ an2^n + b2^n + c \text{ to make the Induction work} \]

(d) Derive the constants.

**Solution:** The first two terms above show that \( a = a/2 + 1 \) and \( b = (b-a)/2 \).
The first equation above implies \( a = 2 \), then the second equation above implies \( b = -2 \).
Using these two values with the base case implies \( c = 2 \). The final result is then
\[
\sum_{k=1}^{n} k2^k = 2n2^n - 2 \cdot 2^n + 2 = 2(n2^n - 2^n + 1) .
\]
5. (15 points) Consider the recurrence

\[ r_n = 4r_{n-1} + r_{n-2} \]

where \( r_1 = 1 \) and \( r_2 = 3 \). We will use Constructive Mathematical Induction to derive an upper bound for \( r_n \). Assume that \( r_n \leq ab^n \). We would primarily like to upper bound \( b \) as tightly as possible, and secondarily upper bound \( a \) as tightly as possible.

(a) What do we learn from the base cases?

**Solution:**

\[
\begin{align*}
1 &= r_1 \leq ab^1 = ab \\
3 &= r_2 \leq ab^2
\end{align*}
\]

(b) What is the Inductive Hypothesis?

**Solution:** Assume that for \( k < n \), \( r_k \leq ab^k \).

(c) Show the Inductive Step.

**Solution:**

\[
\begin{align*}
 r_n &= 4r_{n-1} + r_{n-2} \\
 &\leq 4ab^{n-1} + ab^{n-2} \quad \text{by IH} \\
 &\leq ab^n \quad \text{to make the Induction work}
\end{align*}
\]

Dividing by \( ab^{n-2} \) and rearranging gives

\[
b^2 - 4b - 1 \geq 0
\]

(d) Derive the constants.

**Solution:** Solving for \( b \) gives

\[
b \geq 2 + \sqrt{5}.
\]

Substituting into the first base case (which dominates the second base case) gives

\[
a \geq \frac{1}{2 + \sqrt{5}} = \frac{1}{b}.
\]

Thus,

\[
r_n \leq (2 + \sqrt{5})^{n-1}.
\]
6. (25 points) A full ternary tree is a tree in which every node has exactly one or three children.

(a) Guess a equation relating the number of internal nodes and leaves in a full ternary tree.

**Solution:**

\[ L = 2I + 1 \]

(b) Give a Non-Structural Mathematical Induction proof of the equation.

i. What is the Base Case?

**Solution:** One node: \( L = 1 \) and \( I = 0 \). Then \( 2I + 1 = 2 \cdot 0 + 1 = 1 = L \).

ii. What is the Inductive Hypothesis?

**Solution:** Assume that for all trees with fewer than \( L \) leaves:

\[ L' = 2I' + 1 \]

iii. What is the Inductive Step?

**Solution:** Let \( T \) be a tree with \( L \) leaves. Find an internal node for which all of its children are leaves. Remove the three children. The new tree has \( L - 2 \) leaves (having lost three but gained one) and \( I - 1 \) internal nodes (having lost one). By the IH, \( L - 2 = 2(I - 1) + 1 \), which implies \( L = 2I + 1 \).
(c) Give a recursive definition of a full trinary tree.

**Solution:** A *full trinary tree* is either a single node, called the root; or a single node, called the root, with three children, each of which is the root of a full trinary tree.

(d) Give a Structural Induction proof of the equation, using the definition from Part (c).

i. What is the Base Case:

**Solution:** One node: \( L = 1 \) and \( I = 0 \). Then \( 2I + 1 = 2 \cdot 0 + 1 = 1 = L \).

ii. What is the Inductive Hypothesis?

**Solution:** Assume that for all trees with fewer than \( L \) leaves:

\[ L' = 2I' + 1 \]

iii. What is the Inductive Step?

**Solution:** Consider a full trinary tree with more than one node. By the recursive definition it consists of a root with three children each of which is a full trinary tree. Call them \( T_A, T_B, \) and \( T_C \). Then \( L = L_A + L_B + L_C \), and \( I = I_A + I_B + I_C + 1 \). By the IH, \( L_A = 2I_A + 1 \), \( L_B = 2I_B + 1 \), and \( L_C = 2I_C + 1 \). Summing these last three equations gives:

\[
L_A + L_B + L_C = (2I_A + 1) + (2I_B + 1) + (2I_C + 1) = 2(I_A + I_B + I_C) + 3
\]

By substitution

\[
L = 2(I - 1) + 3 = 2I + 1.
\]