Set Theory
Set definitions

Definition: A set is a collection of objects.

Examples: \( A = \{1,2,3\} \)

\[
B = \{ x \in \mathbb{Z} \mid -4 < x < 4 \}
\]

\[
C = \{ x \in \mathbb{Z}^+ \mid -4 < x < 4 \}
\]

A set is completely defined by its elements, i.e.,
\[
\{a,b\} = \{b,a\} = \{a,b,a\} = \{a,a,a,b,b,b\}
\]
More set concepts

- The universal set $U$ is the set of all elements under consideration
- A set can be finite or can be infinite
- For a set $S$, $n(S)$ or $|S|$ are used to refer to the cardinality of $S$, which is the number of elements in $S$
- The symbol $\in$ means "is an element of"
- The symbol $\notin$ means "is not an element of"
Subset

- $A \subseteq B \iff (\forall x \in U) [x \in A \rightarrow x \in B]$
  
  - $A$ is contained in $B$
  - $B$ contains $A$

- $A \not\subseteq B \iff (\exists x \in U) [x \in A \land x \notin B]$

- Relationship between membership and subset:
  
  $(\forall x \in U) [x \in A \iff \{x\} \subseteq A]$

- Definition of set equality: $A = B \iff A \subseteq B \land B \subseteq A$
Proper subset

\[ A \subset B \iff A \subseteq B \land A \neq B \]
Do these represent the same sets or not?

\[ X = \{x \in \mathbb{Z} \mid (\exists p \in \mathbb{Z})[x = 2p]\} \]
\[ Y = \{y \in \mathbb{Z} \mid (\exists q \in \mathbb{Z})[y = 2q - 2]\} \]

\[ A = \{x \in \mathbb{Z} \mid (\exists i \in \mathbb{Z})[x = 2i + 1]\} \]
\[ B = \{x \in \mathbb{Z} \mid (\exists i \in \mathbb{Z})[x = 3i + 1]\} \]
\[ C = \{x \in \mathbb{Z} \mid (\exists i \in \mathbb{Z})[x = 4i + 1]\} \]
Formal definitions of set operations

Union: \[ A \cup B = \{ x \in A \lor x \in B \} \]

Intersection: \[ A \cap B = \{ x \in A \land x \in B \} \]

Complement: \[ A^c = A' = \overline{A} = \{ x \in U \mid x \notin A \} \]

Difference: \[ A - B = \{ x \in A \land x \notin B \} \]
\[ A - B = A \cap B \]
Venn Diagrams

Sets are represented as regions to illustrate relationships between them.
The empty set $∅$ has no elements, so $∅ = \{\}$. 

1. $(\forall \text{ sets } X)[∅ \subseteq X]$ 
2. There is only one empty set. 
3. $(\forall \text{ sets } X)[X \cup ∅ = X]$ 
4. $(\forall \text{ sets } X)[X \cap X = ∅]$ 
5. $(\forall \text{ sets } X)[X \cap ∅ = ∅]$ 
6. $U = ∅$ 
7. $∅ = U$
The Cartesian Product

- The Cartesian product of sets $A$ and $B$ is defined as
  \[ A \times B = \{(a, b) \mid a \in A \land b \in B\} \]

- \[ |A \times B| = |A| \cdot |B| \quad \text{(size of } A \text{ times size of } B) \]
Ordered n-tuples

- An ordered $n$-tuple takes order and multiplicity into account

- The tuple $(x_1, x_2, x_3, \ldots, x_n)$
  - has $n$ values
  - not necessarily distinct
  - order matters (unlike sets)

- $(x_1, x_2, x_3, \ldots, x_n) = (y_1, y_2, y_3, \ldots, y_n) \iff (\forall i \in 1 \leq i \leq n)[x_i = y_i]$

- 2-tuples are called pairs, and 3-tuples are called triples
Examples

- \((1,3,2) \neq (3,2,1)\)
- \(\{1,3,2\} = \{3,2,1\}\)
- \((1,1,3,2)\) is a fine 4-tuple
- \((1,1,3,2) \neq (1,3,2)\)
- \(\{1,1,3,2\} = \{1,3,2\}\)
Disjoint Sets

- $A$ and $B$ are disjoint

  $\iff A$ and $B$ have no elements in common

  $\iff (\forall x \in U)[x \in A \rightarrow x \notin B]$

- $A \cap B = \emptyset \iff A$ and $B$ are disjoint sets

- If $A$ and $B$ are disjoint then $|A \cup B| = |A| + |B|

  What if $A$ and $B$ are not disjoint?
Power set

$P(A) = \text{the set of all subsets of } A$

Examples: What are

$P\{a,b\}$?
$P\{a,b,c\}$?
$P\{a\}$?
$P(\emptyset)$?
$P\{\emptyset\}$?
Properties of power sets

- $\forall$ sets $A, B$ [$A \subseteq B \rightarrow P(A) \subseteq P(B)$]

- $\forall$ sets $A$ [$|P(A)| = 2^{|A|}$]

Proof in class
Properties of sets

- **Inclusion**
  \[ A \cap B \subseteq A \quad A \cap B \subseteq B \]
  \[ A \subseteq A \cup B \quad B \subseteq A \cap B \]

- **Transitivity**
  \[ A \subseteq B \land B \subseteq C \rightarrow A \subseteq C \]
More Properties of Sets

- DeMorgan’s for complement
  \[(A \cap B) = \overline{A} \cup \overline{B}\]
  \[(A \cup B) = \overline{A} \cap \overline{B}\]

- Distribution of union and intersection
  \[A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\]
  \[A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\]

- There are a number of others
Analogies between Sets and Logic

- $A \cup B$ is analogous to $A \lor B$
- $A \cap B$ is analogous to $A \land B$
- $\overline{A}$ is analogous to $\neg A$
- What is $A \rightarrow B$ analogous to?
Using Venn diagrams to help find counterexamples

\[ A \cap (B \cap C) = ? = (A \cap B) \cap (A \cap C) \]

\[ A \cap (B \cap C) = ? = (A \cap B) \cap C \]
Deriving new properties using Venn diagrams

\[ B - (A \cap C) = (B - A) \triangle (B - C) \]

\[ A - B = A - (A \cap B) \]

\[ A \subseteq B \land A \subseteq C \rightarrow A \subseteq (B \cap C) \]

Proof in class
Partitions of a Set

- A collection of nonempty sets \( \{A_1, A_2, \ldots, A_n\} \) is a partition of the set \( A \) if and only if
  1. \( A = A_1 \cup A_2 \cup \ldots \cup A_n \)
  2. \( A_1, A_2, \ldots, A_n \) are mutually disjoint

- An infinite set can be partitioned. The partitions can be infinite, or can be finite.
Russell’s Paradox

- A set can be an element or member of itself.
  Example: List of all lists.
    Set of infinite sets.

- Consider the set $S = \{A \mid A$ is a set and $A \notin A\}$
  - Is $S$ an element of itself?
Russell’s Paradox Continued

- $S = \{A \mid A \text{ is a set and } A \notin A\}$
  - Is $S$ an element of itself?
  - Option 1: YES! $S \in S$. Then by def of $S$, $S \notin S$.
  - Option 2: NO! $S \notin S$. Then by def of $S$, $S \in S$.
  - Upshot: $S$ does not exist!!!
The Halting Problem

Does such a computer program exist?

INPUT: Program $P$, input $x$

OUTPUT: YES if program $P$ on input $x$ halts

NO if program $P$ on input $x$ does not halt

An approach: Run $P$ on $x$ and see what happens.

Does not work!: If it does not halt we will never know.

How about a different approach???
Proof for the Halting Problem

Suppose there is a program $\text{Halt}(P,x)$ for the halting problem.

Create a new program $\text{Test}$:

```plaintext
procedure Test(P)
    if Halt(P,P) outputs YES then loop forever;
    if Halt(P,P) outputs NO then STOP;
end procedure
```

Now run $\text{Test(Test)}$:

- if $\text{Test(Test)}$ halts, then $\text{Halt(Test,Test)}$ outputs YES; hence $\text{Test(Test)}$ does not halt.
- if $\text{Test(Test)}$ does not halt, then $\text{Halt(Test,Test)}$ outputs NO; hence $\text{Test(Test)}$ halts.