WAIT FOR INSTRUCTIONS BEFORE BEGINNING

HONOR PLEDGE: "I pledge on my honor that I have not given or received any unauthorized assistance on this examination."

Signature and UID:

PRINT Name:

- Write your answers with enough detail about your approach and concepts used, so that the grader will be able to understand it easily.
- The sum of the grades is 105, but your grades would be out of 100 (thus you get 5 bonus points by solving all the problems).
- In this exam, \( n \) denotes the number of vertices and \( m \) denotes the number of edges.

Select the best choice for the first 8 problems and mark it by \( X \) in the table below:

<table>
<thead>
<tr>
<th>Problem</th>
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DO NOT WRITE BELOW THIS LINE

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<table>
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<tr>
<td>Problems 1-8</td>
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<td>Problem 9</td>
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<td>Problem 11</td>
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Problem 1 (5 points) Run Dijkstra’s algorithm on the weighted graph below, using vertex $A$ as the source. Write the vertices in the order which they are marked.

![Graph](image1.png)

a) $A, B, E, C, D, G, H, F$

b) $A, B, E, C, D, H, G, F$

c) $A, B, E, F, G, C, D, H$

d) $A, E, B, C, D, G, H, F$

e) $A, B, C, D, E, G, F, H$

Problem 2 (5 points) Run Prim’s algorithm on the weighted graph below, using vertex $A$ as the source. What is the sum of the weights of the first, the third, and the fifth edges that are added to the output of Prim’s algorithm?

![Graph](image2.png)

a) 9

b) 10

c) 11

d) 12

e) 13

Problem 3 (5 points) Find an optimal parenthesization of $A_1 \times A_2 \times A_3 \times A_4$ where $A_1$ is a $2 \times 4$ matrix, $A_2$ is a $4 \times 5$ matrix, $A_3$ is a $5 \times 3$ matrix, and $A_4$ is a $3 \times 2$ matrix. What is the cost of an optimal solution?

a) 78

b) 82

c) 86

d) 90

e) 96

Problem 4 (5 points) Consider an execution of the DFS algorithm on directed graph $G$. Which of the following statements is correct?

(I) Edge $(u, v)$ is a cross-edge iff $v.\text{finish} < u.\text{start}$.

(II) Edge $(u, v)$ is a back-edge iff $u.\text{finish} < v.\text{finish}$.

(III) Edge $(u, v)$ is a tree-edge iff $u.\text{start} < v.\text{start}$ and $u.\text{finish} > v.\text{finish}$.

a) I

b) II

c) III

d) I and II

e) II and III

Problem 5 (5 points) The following procedure computes $2^n$ for any non-negative integer $n$. What is its running time?

```
Algorithm 1 Power(n)
1: if n=0 then
2:   return 1
3: end if
4: return Power(n-1) + Power(n-1)
```

a) $O(\log n)$

b) $O(n)$

c) $O(n \log n)$

d) $O(n^2)$

e) $O(2^n)$
Problem 6 (5 points) For a graph with $\Theta(n)$ weighted edges, what is the best running time for Dijkstra’s algorithm?
   a) $O(n)$  b) $O(n \log n)$  c) $O(n \log^2 n)$  d) $O(n^2)$  e) $O(n^2 \log n)$

Problem 7 (5 points) Let the following tree be a BFS tree of a directed graph $G$. Which of the following edges cannot be in graph $G$?
   a) $(D, A)$  b) $(B, C)$  c) $(A, E)$  d) $(B, G)$  e) $(F, C)$

Problem 8 (5 points) Consider the following as the output of the DFS algorithm on an undirected graph. Which of the following edges is not in the corresponding DFS tree?

<table>
<thead>
<tr>
<th>Vertex</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<tr>
<td>Start Time</td>
<td>1</td>
<td>2</td>
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<td>11</td>
<td>6</td>
<td>8</td>
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<tr>
<td>Finish Time</td>
<td>14</td>
<td>13</td>
<td>10</td>
<td>4</td>
<td>12</td>
<td>7</td>
<td>9</td>
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</tbody>
</table>

   a) $(B, C)$  b) $(B, D)$  c) $(C, G)$  d) $(B, G)$  e) $(A, B)$
Problem 9 (25 points) Answer the following questions regarding Kruskal’s algorithm for finding a minimum spanning tree.

(a) (5 points) Describe briefly Kruskal’s algorithm?

(b) (5 points) Assume procedures FIND(u) and UNION(u, v) are given. Write the pseudo-code for Kruskal’s algorithm.

(c) (5 points) Write pseudo-codes for both FIND(u) and UNION(u, v) which run in $O(\log n)$.

(d) (10 points) Prove the correctness of the algorithm.

Solution: Look at Chapter 23 of the book by Cormen et al.
Problem 10 (20 points) Matrix $M$ with $n$ rows and $m \geq n$ columns is given. Let cell $(1, 1)$ be the lower left corner of the matrix. We want to choose exactly one cell from each row such that:

- For every $1 \leq k < n$, the selected cell from the $k$-th row is on the left of the selected cell from the $k + 1$-th row.
- The sum of all selected cells is maximized.

The goal is to design a dynamic program which runs in $O(nm)$ for solving this problem. For every $i \leq n$ and $j \leq m$, consider submatrix $M'$ with the first $i$ rows and the first $j$ columns of matrix $M$, and let $A[i, j]$ be the optimum solution for submatrix $M'$.

(a) (10 points) Consider a decision regarding cell $(i, j)$. Give a formula for computing $A[i, j]$ based on the solutions of subproblems.

Solution: Here is the formula:

$$A[i, j] = \max\{A[i, j - 1], A[i - 1, j - 1] + M[i, j]\}$$

(b) (3 points) What are base cases? What would be their values?

Solution: Here are the base cases:

$$\forall 0 \leq j \leq m, \ A[0, j] = 0$$

$$\forall 1 \leq i \leq n, \ A[i, 0] = -\infty$$

(c) (7 points) Write a pseudo-code for solving the problem?

Solution:

```
Algorithm 2 Dynamic(M, n, m)
1: for i = 1 to n do
2: \hspace{1em} A[i, 0] = -\infty
3: end for
4: for j = 0 to m do
5: \hspace{1em} A[0, j] = 0
6: end for
7: for i = 1 to n do
8: \hspace{2em} for j = 1 to m do
9: \hspace{3em} A[i, j] = \max\{A[i, j - 1], A[i - 1, j - 1] + M[i, j]\}
10: \hspace{2em} end for
11: end for
12: return A[n, m]
```
Problem 11 (20 points) Consider the following implementation of the Floyd-Warshall algorithm. Assume $w_{ij} = \infty$ when there is no edge between vertex $i$ and vertex $j$, and assume $w_{ii} = 0$ for every vertex $i$.

Algorithm 3 Floyd-Warshall($G$)

\begin{verbatim}
1: for $i = 1$ to $n$ do
2:   for $j = 1$ to $n$ do
3:     $A[i, j, 0] = w_{ij}$
4:     $P[i, j] = -1$
5:   end for
6: end for
7: for $k = 1$ to $n$ do
8:   for $i = 1$ to $n$ do
9:     for $j = 1$ to $n$ do
13:       $P[i, j] = k$
14:    end if
15:   end for
16: end for
17: end for
\end{verbatim}

(a) (12 points) Assume matrix $P$, the output of the above algorithm, is given. Design an algorithm for finding the shortest path from $u$ to $v$ by using matrix $P$, and write its pseudo-code.

Solution:

Algorithm 4 Solution($u, v, P$)

\begin{verbatim}
1: if $u == v$ then
2:   return $\{u\}$
3: end if
4: return $\{u\} + \text{Find-Path}(u, v, P) + \{v\}$
\end{verbatim}

Algorithm 5 Find-Path($u, v, P$)

\begin{verbatim}
1: if $P[u, v] == -1$ then
2:   return $\emptyset$
3: end if
4: $k = P[u, v]$
5: return $\text{Find-Path}(u, k, P) + \{k\} + \text{Find-Path}(k, v, P)$
\end{verbatim}

(b) (8 points) Consider the following matrix as matrix $P$ for graph $G$ with 6 vertices. What is the shortest path from vertex 1 to vertex 2 in graph $G$? What is the shortest path from vertex 5 to vertex 7 in graph $G$?

Solution: The shortest path from vertex 1 to 2 is 1, 4, 3, 5, 2. The shortest path from vertex 5 to 7 is 5, 3, 6, 7.
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<th>$P$</th>
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