Linear Algebra Review

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Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n\times m} = A_{n\times m} + B_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m} \qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad (AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Matrices

Determinant: A must be square

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $$\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$$

Matrices

Inverse: A must be square

$$A_{n\times n}A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n}A_{n\times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

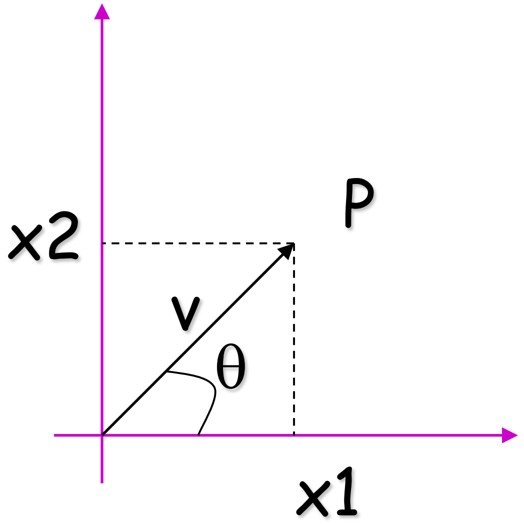
Example:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} . \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} . \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D Vector

$$\mathbf{v} = (x_1, x_2)$$



Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^{\,2} + x_2^{\,2}}$

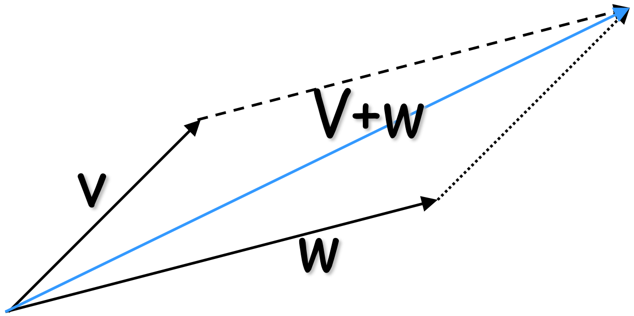
If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ Is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right)$$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left(\frac{x_2}{x_1} \right)$

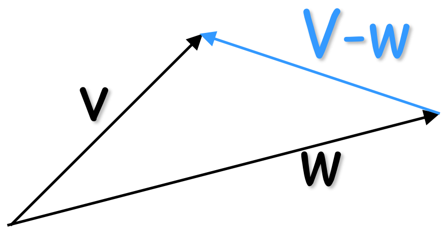
Vector Addition

$$\mathbf{v}+\mathbf{w}=(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)$$



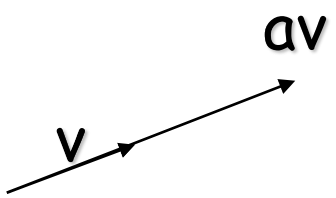
Vector Subtraction

$$\mathbf{v}-\mathbf{w}=(x_1,x_2)-(y_1,y_2)=(x_1-y_1,x_2-y_2)$$



Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



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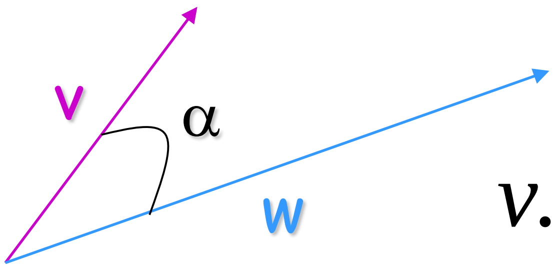
Inner (dot) Product



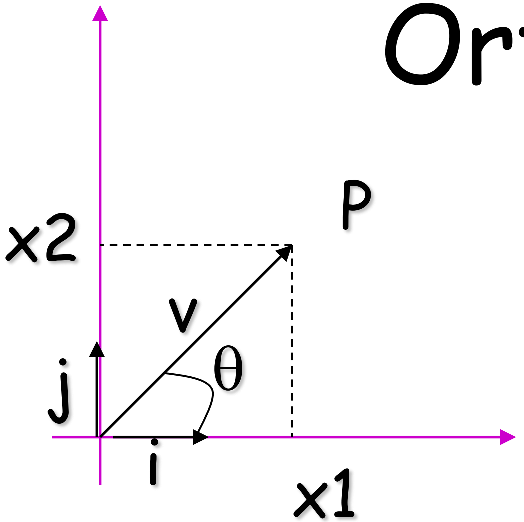
$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2.y_2$$

The inner product is a SCALAR!

$$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

Orthonormal Basis



$$\mathbf{i} = (1,0) \qquad \|\mathbf{i}\| = 1$$

$$\mathbf{j} = (0,1) \qquad \|\mathbf{j}\| = 1$$

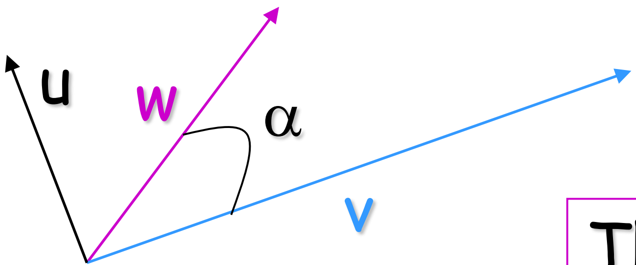
$$\mathbf{i} \cdot \mathbf{j} = 0$$

$$\mathbf{v} = (x_1, x_2) \qquad \mathbf{v} = x_1.\mathbf{i} + x_2.\mathbf{j}$$

$$\mathbf{v}.\mathbf{i} = (x_1.\mathbf{i} + x_2.\mathbf{j}).\mathbf{i} = x_1.1 + x_2.0 = x_1$$

$$\mathbf{v}.\mathbf{j} = (x_1.\mathbf{i} + x_2.\mathbf{j}).\mathbf{j} = x_1.0 + x_2.1 = x_2$$

Vector (cross) Product



$$u = v \times w$$

The cross product is a VECTOR!

Magnitude: $\| \mathrm{u} \| = \| v.w \| = \| v \| \| w \| \sin \alpha$

Orientation:
$$u \perp v \Rightarrow u \cdot v = (v \times w) \cdot v = 0$$
$$u \perp w \Rightarrow u \cdot w = (v \times w) \cdot w = 0$$

Vector Product Computation

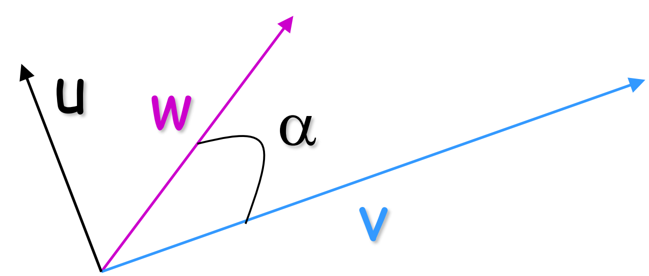
$$\mathbf{k} = (0,0,1) \qquad \|\mathbf{k}\| = 1$$

$$\mathbf{j} = (0,1,0) \qquad \|\mathbf{j}\| = 1 \qquad \mathbf{i}\cdot\mathbf{j} = 0,\ \mathbf{i}\cdot\mathbf{k} = 0,\ \mathbf{j}\cdot\mathbf{k} = 0$$

$$\mathbf{i} = (1,0,0) \qquad \|\mathbf{i}\| = 1$$

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

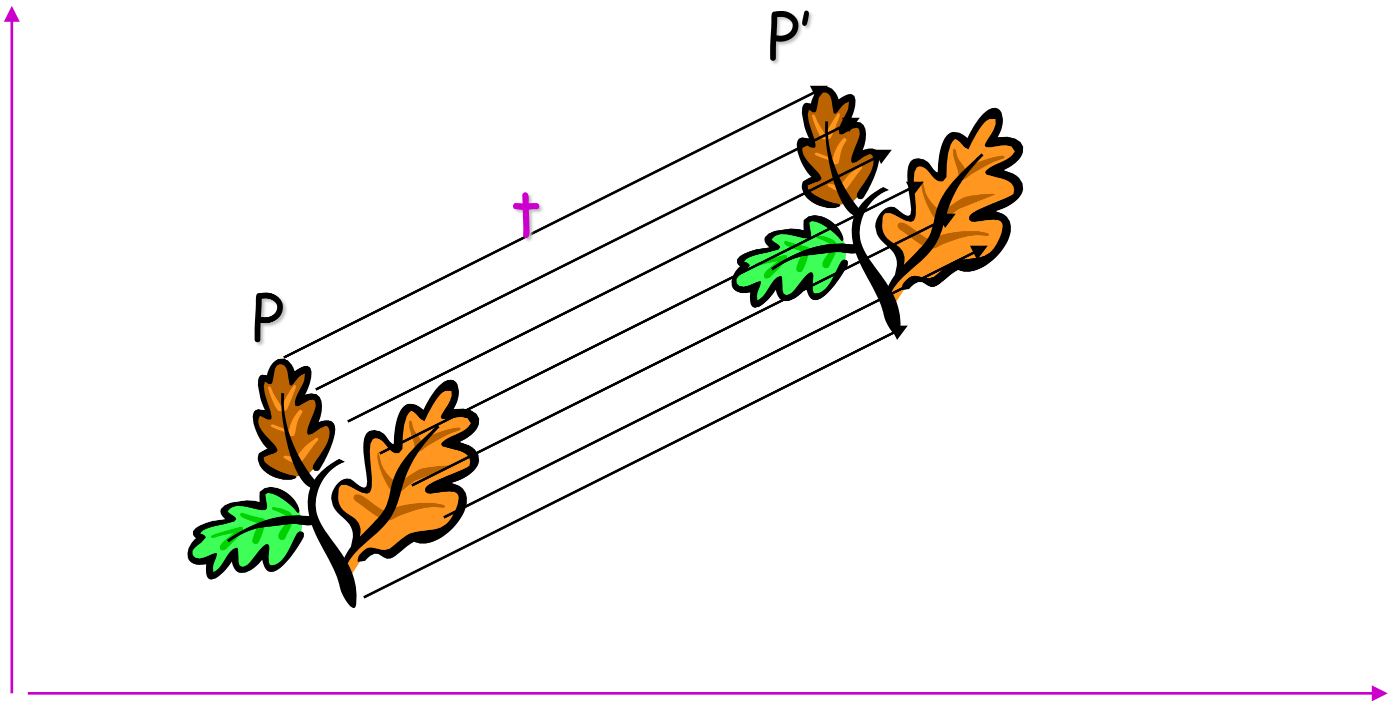
$$\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$



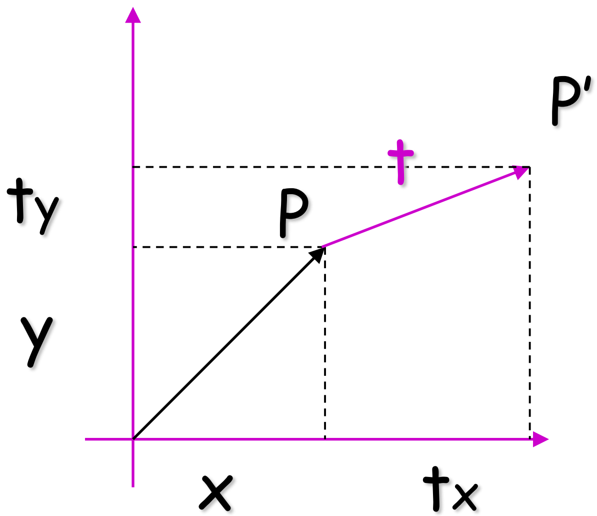
$$= (x_2 y_3 - x_3 y_2)\mathbf{i} + (x_3 y_1 - x_1 y_3)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

2D Geometrical Transformations

2D Translation



2D Translation Equation

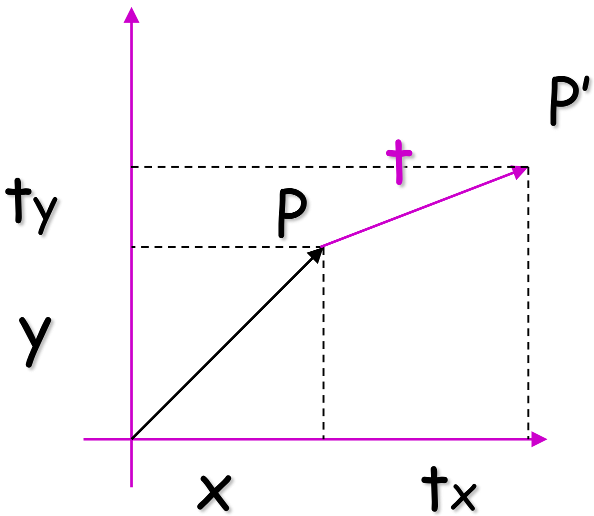


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$

2D Translation using Matrices



$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

t P

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \rightarrow (x \cdot z, y \cdot z, z) \quad z \neq 0$$

$$(x, y, z) \rightarrow (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

- NOTE: If the scalar is 1, there is no need for the multiplication!

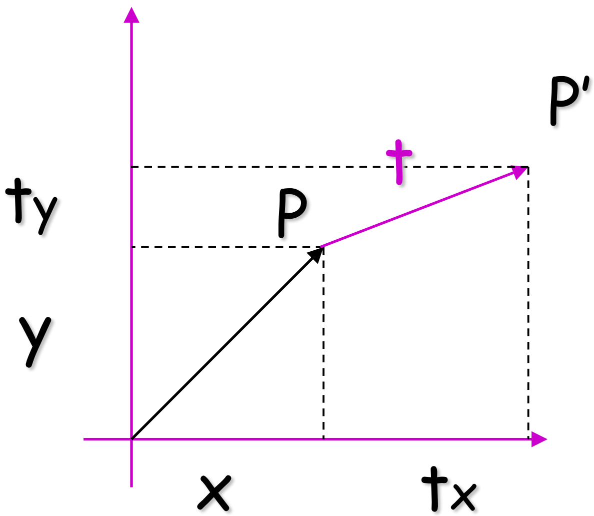
Back to Cartesian Coordinates:

- Divide by the last coordinate and eliminate it. For example,

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$

$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

2D Translation using Homogeneous Coordinates



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

t

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

P

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{T} \cdot \mathbf{P}$$

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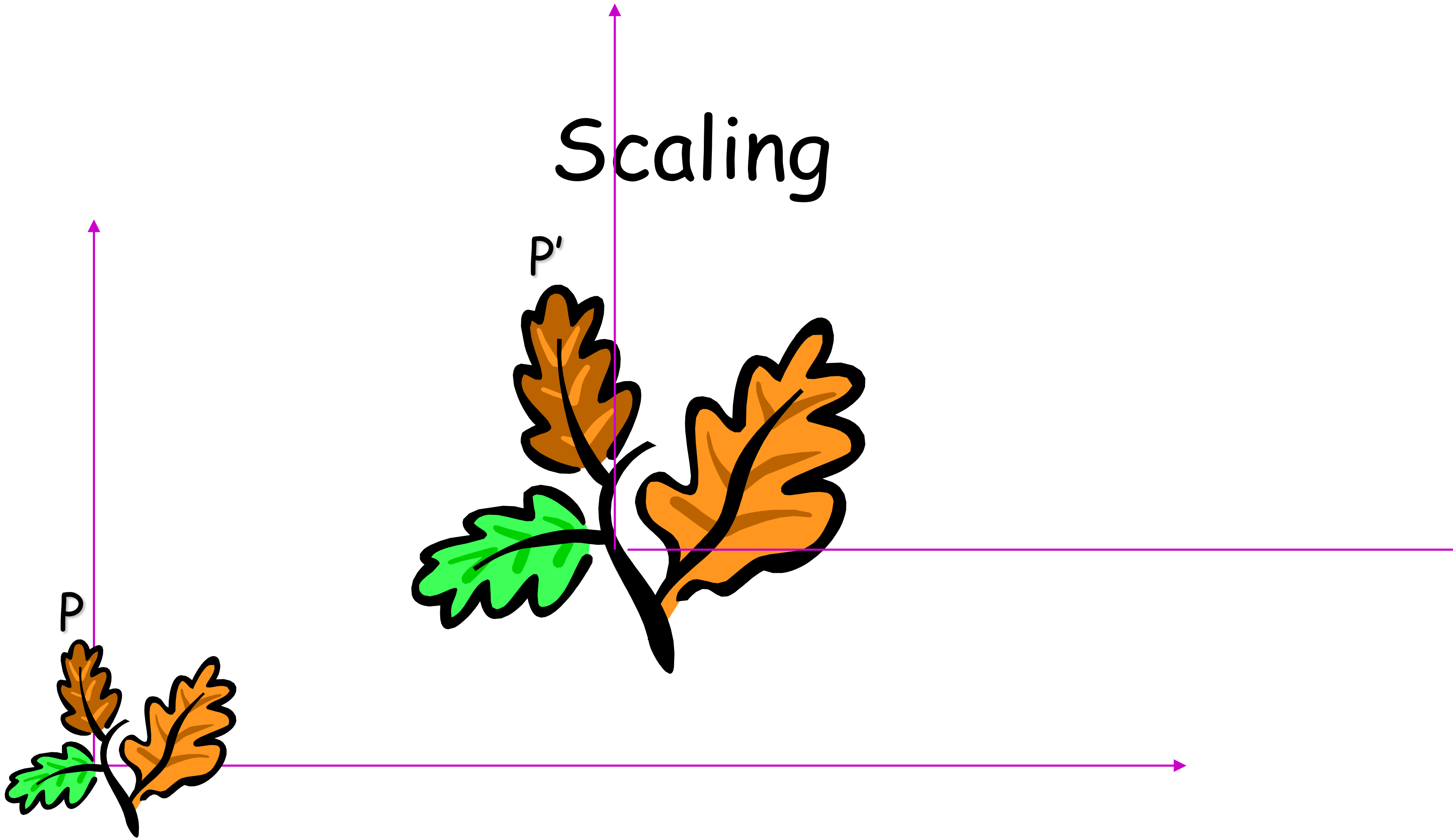
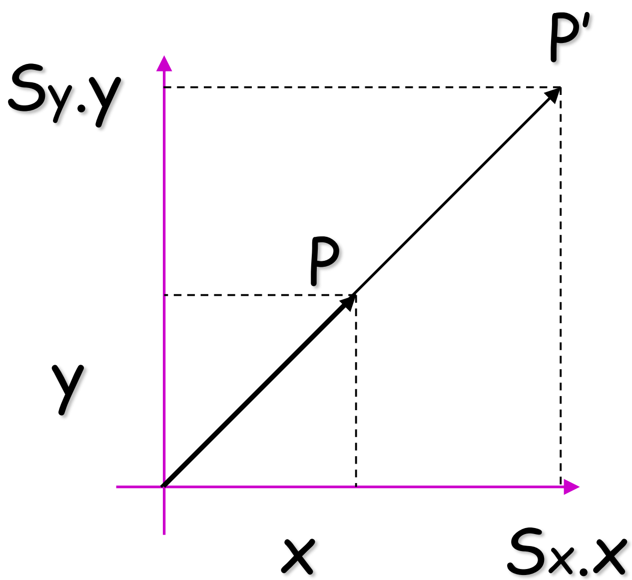
Scaling



Scaling Equation



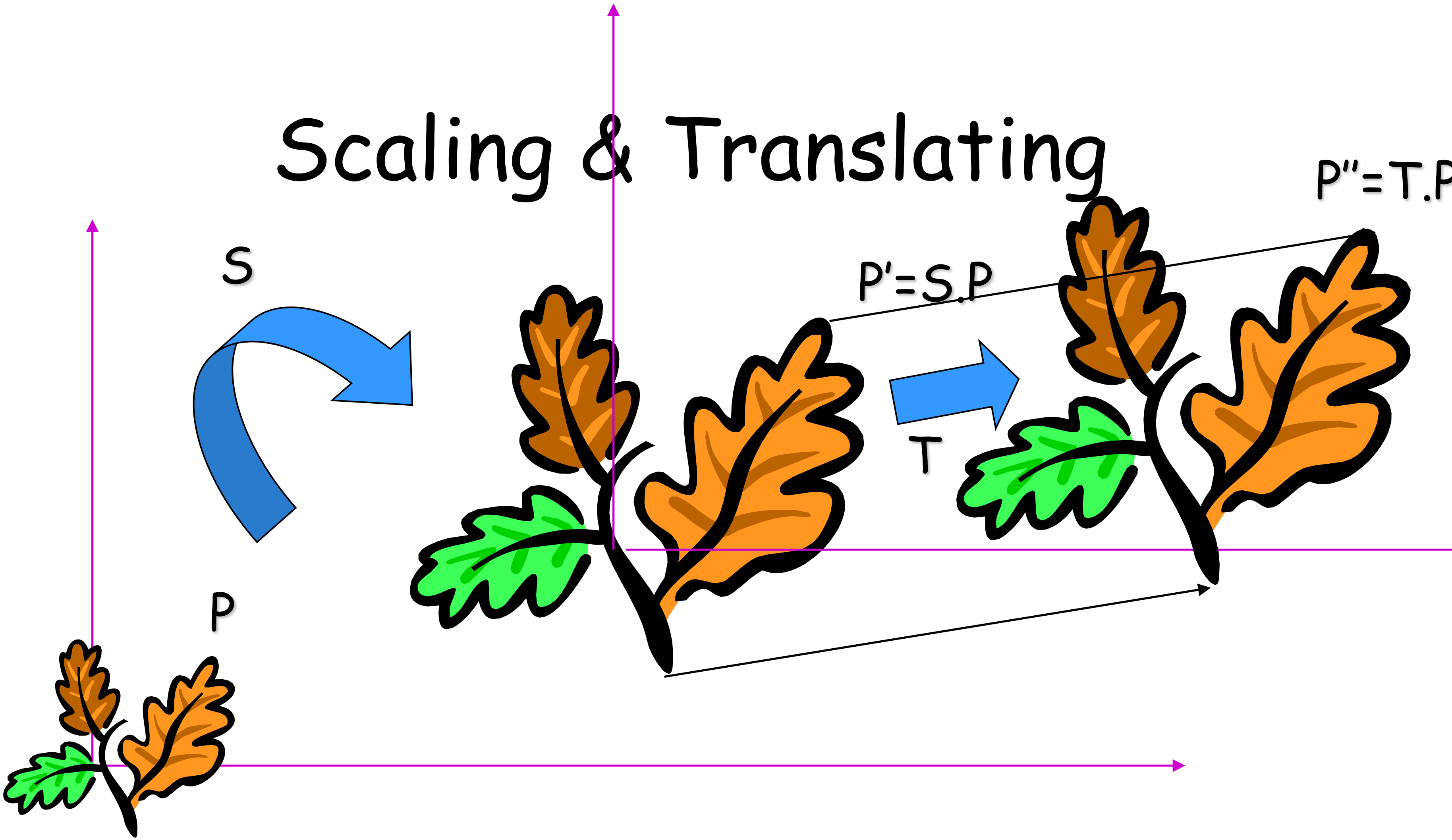
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



P''=T.P'=T.(S.P)=(T.S).P

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Scaling & Translating

P''=T.P'=T.(S.P)=(T.S).P

$$\mathbf{P}'' = \mathbf{T}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

Translating & Scaling $\neq$ Scaling & Translating

P''=S.P'=S.(T.P)=(S.T).P

$$\mathbf{P}''=\mathbf{S}\cdot\mathbf{T}\cdot\mathbf{P}=\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=$$

$$=\begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

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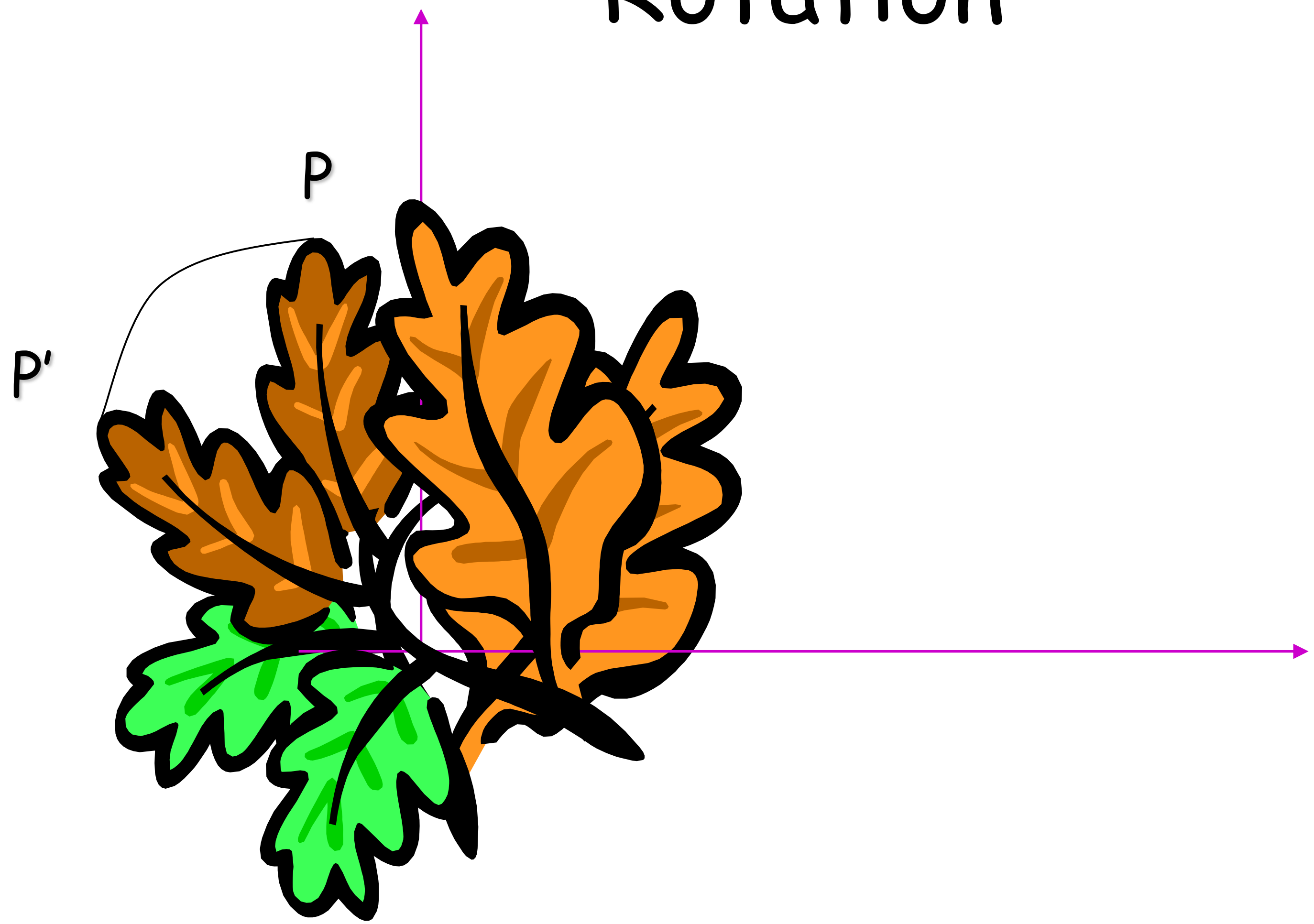
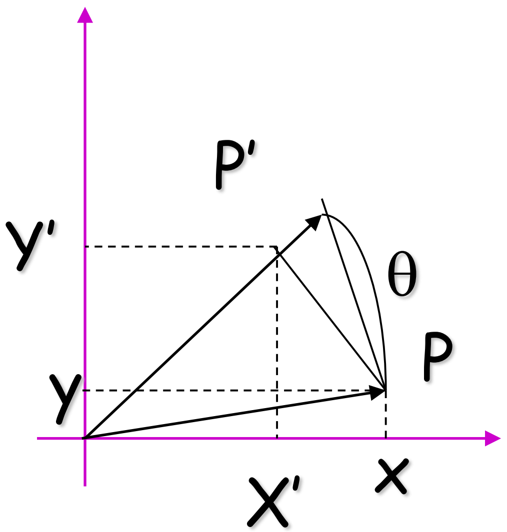
Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P' = R.P}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 $\rightarrow$ 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Scaling, Translating & Rotating

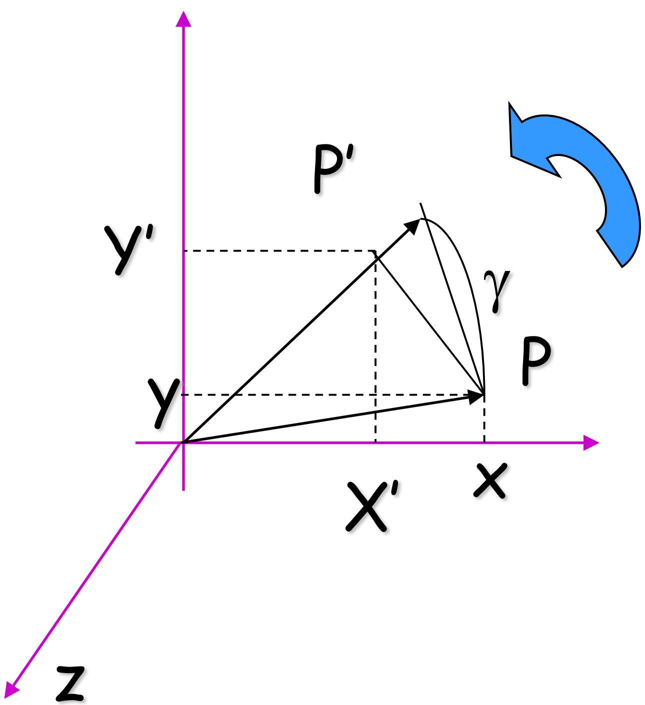
Order matters!

$$P' = S.P$$
$$P'' = T.P' = (T.S).P$$
$$P''' = R.P'' = R.(T.S).P = (R.T.S).P$$

$$R.T.S \neq R.S.T \neq T.S.R \ldots$$

3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

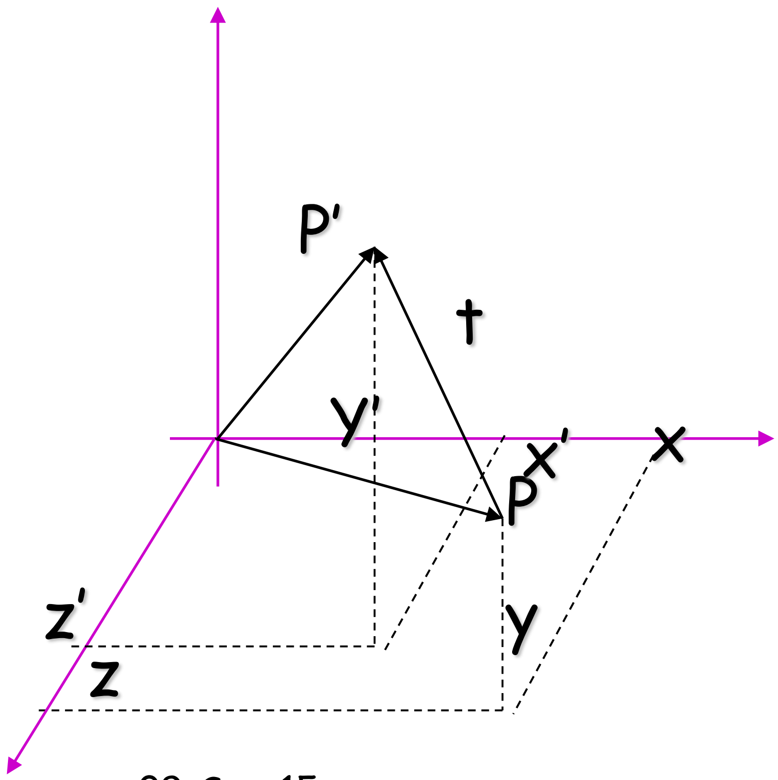
3D Rotation (axis & angle)

$$\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T \quad , \quad \text{angle } \theta$$

$$\mathbf{R} = \mathbf{I}\cos\theta + \mathbf{I}(1-\cos\theta)\begin{bmatrix} n_1^{\ 2} & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^{\ 2} & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^{\ 2} \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

3D Translation of Points

Translate by a vector $t=(t_x,t_y,t_x)^T$:



$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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