Fourier Transform

- Analytic geometry gives a coordinate system for describing geometric objects.
- Fourier transform gives a coordinate system for functions.

Decomposition of the image function

The image can be decomposed into a weighted sum of sinusoids and cosinuoids of different frequency.

Fourier transform gives us the weights

Basis

- P=(x,y) means P = x(1,0)+y(0,1)
- Similarly:

$$f(\theta) = a_{11}\cos(\theta) + a_{12}\sin(\theta)$$

$$+ a_{21}\cos(2\theta) + a_{22}\sin(2\theta) + \ldots$$

$$\forall c, \exists a_1, a_2 \text{ such that}:$$

$$\sin(\theta + c) = a_1 \cos\theta + a_2 \sin\theta$$

$$a_1 = \sin c \qquad \mathrm{a}_2 = \mathrm{cosc}$$

Orthonormal Basis

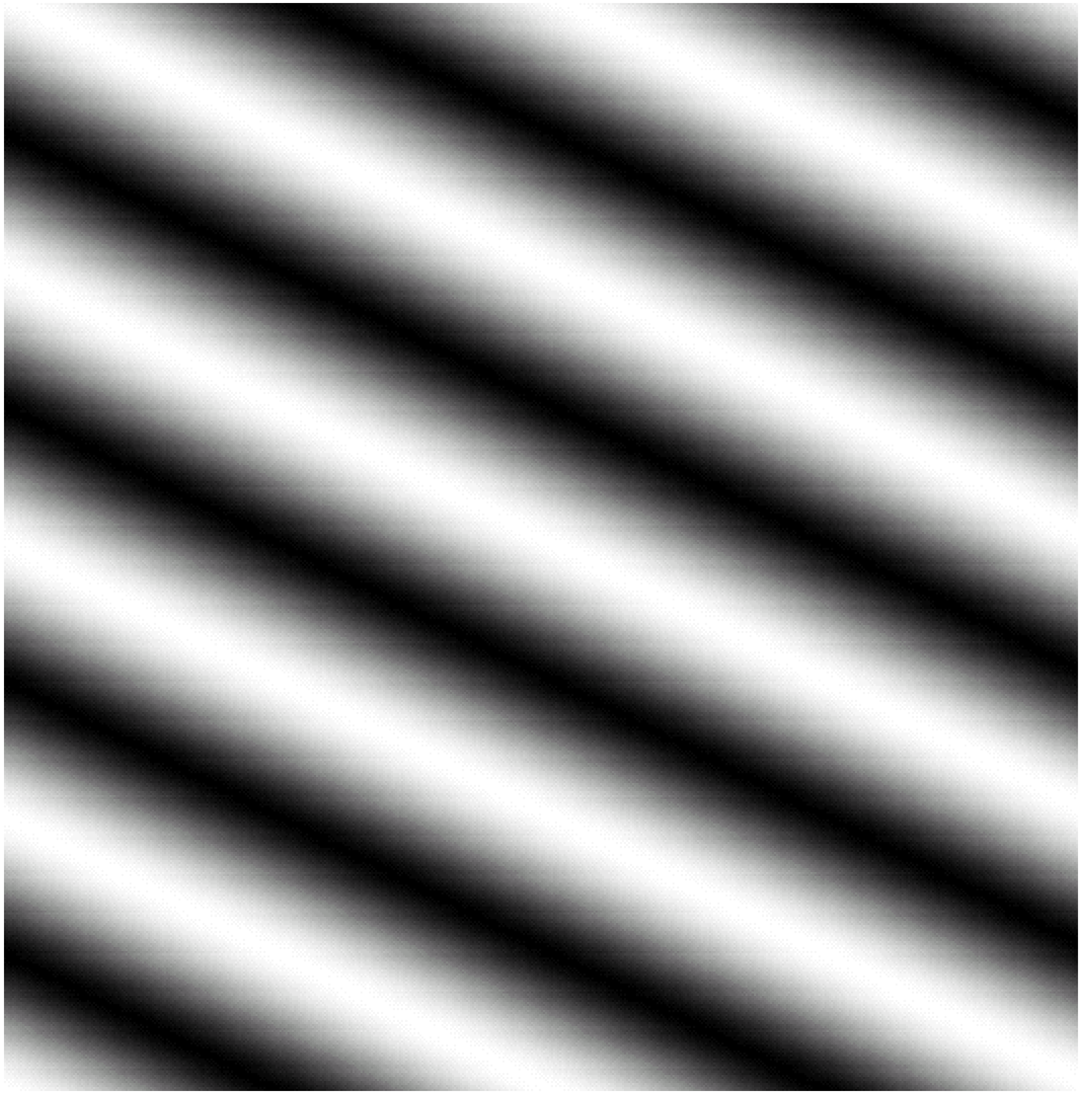
- $\|(1,0)\|=\|(0,1)\|=1$
- $(1,0).(0,1)=0$
- Similarly we use normal basis elements eg:

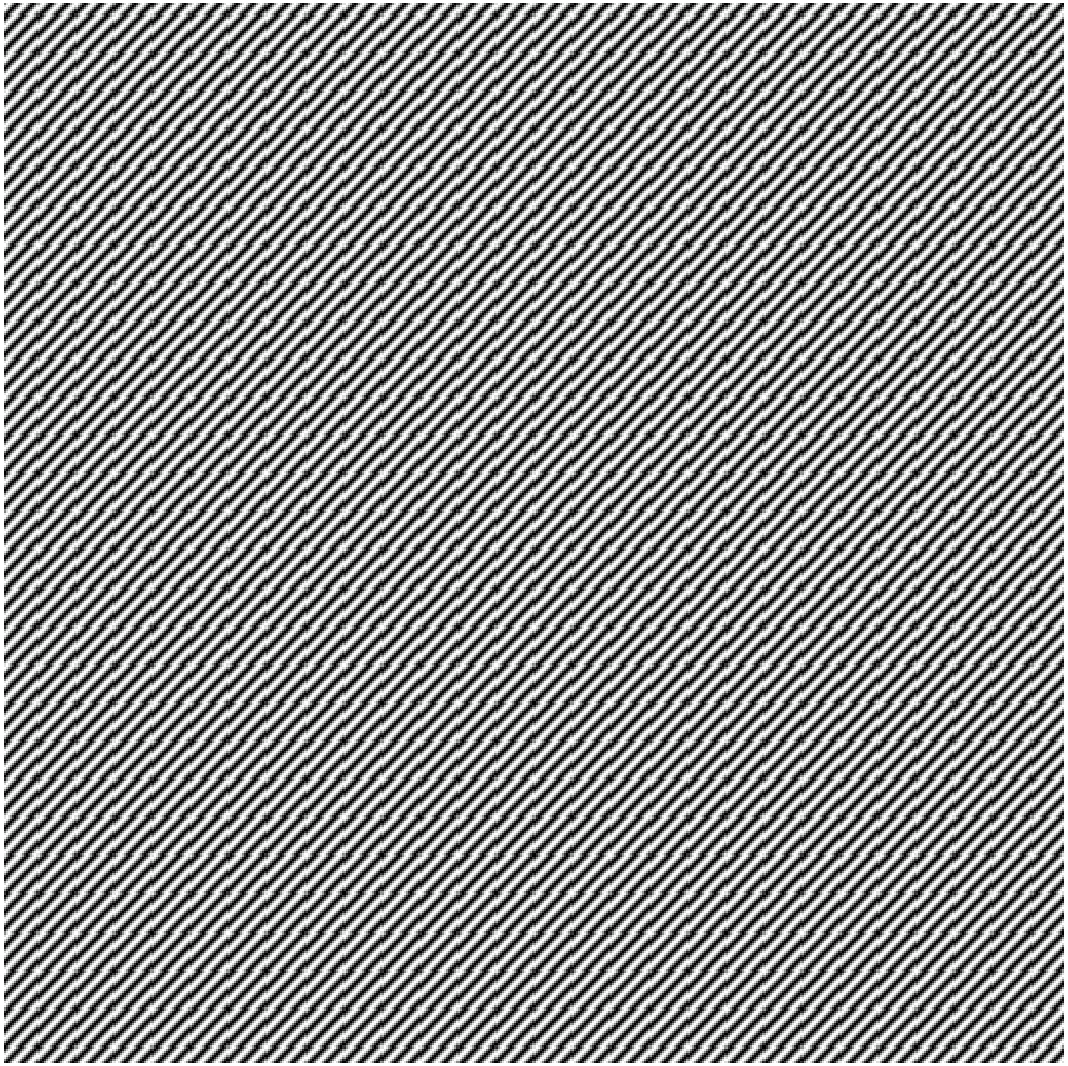
$$\frac{\cos(\theta)}{\|\cos(\theta)\|} \qquad \|\cos(\theta)\| = \sqrt{\int_0^{2\pi} \cos^2\theta \, d\theta}$$

- While, eg:

$$\int_0^{2\pi} \cos\theta \sin\theta \, d\theta = 0$$

2D Example





Why are we interested in a decomposition of the signal into harmonic components?

Sinusoids and cosinuoids are
eigenfunctions of convolution

$$e^{i\omega t} \rightarrow \boxed{} \rightarrow \mathrm{A}(\omega)\mathrm{e}^{\mathrm{i}\omega \mathrm{t}}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

Thus we can understand what the system (e.g filter) does
to the different components (frequencies) of the signal (image)

Convolution Theorem

$$f \otimes g = T^{-1} F * G$$

- *F,G* are transform of *f,g* , T^{-1} *is inverse Fourier transform*

That is, *F* contains coefficients, when we write *f* as linear combinations of harmonic basis.

Fourier transform

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}dxdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cos(ux+vy)dxdy+i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\sin(ux+vy)dxdy=$$

$$\Re(\mathrm{F})+\mathrm{i}\Im(\mathrm{F})$$

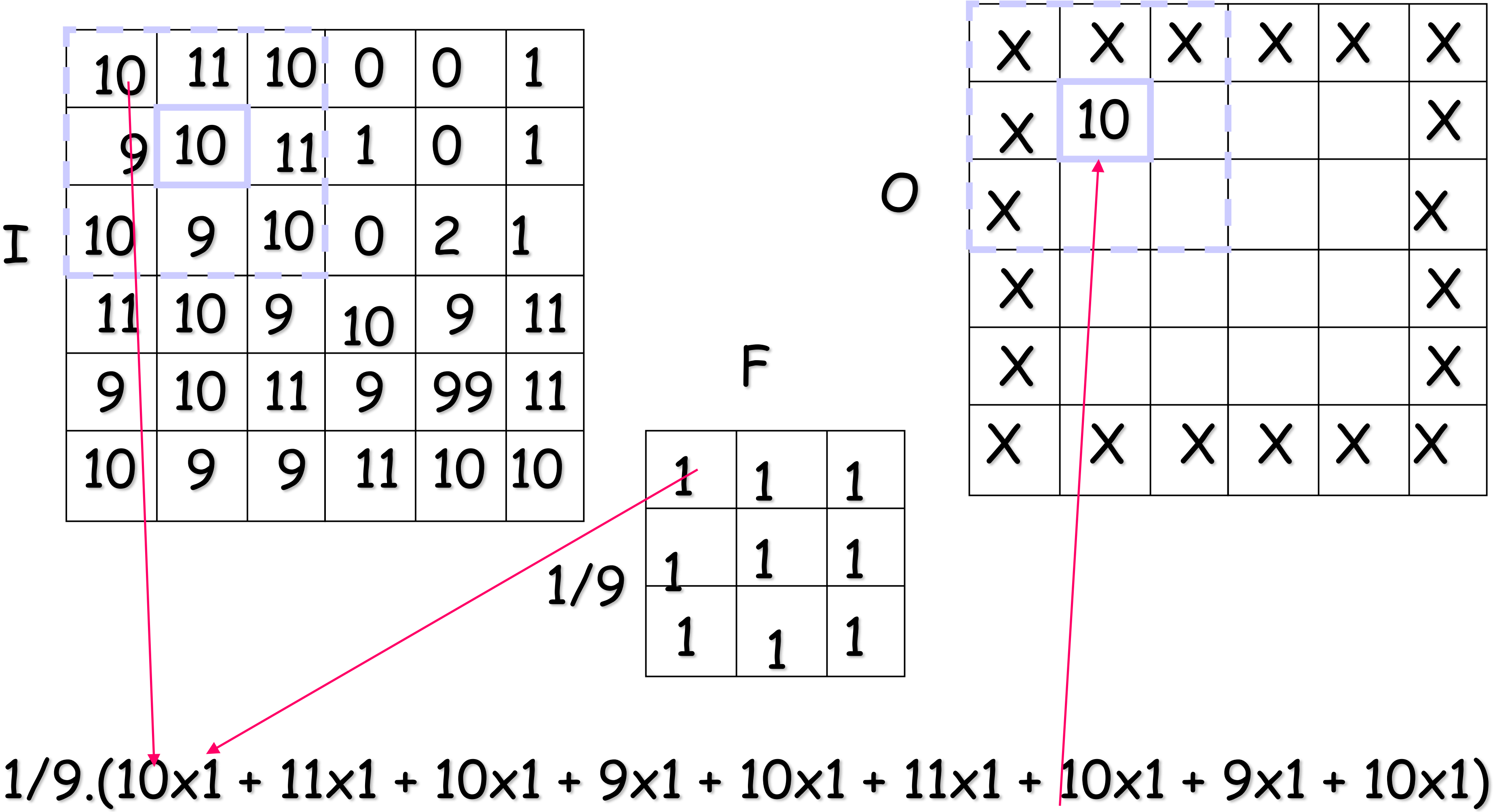
often described by magnitude ($\sqrt{\Re^2(F)+\Im^2(F)}$)

and phase ($\arctan(\frac{\Im(F)}{\Re(F)})$)

In the discrete case with values f_{kl} of f(x,y) at points (kw,lh) for k= 1..M-1, l= 0..N-1

$$F_{mn}=\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f_{kl}e^{-\pi i(\frac{k\,m}{M}+\frac{l\,n}{N})}$$

Remember Convolution



I

10	11	10	0	0	1
9	10	11	1	0	1
10	9	10	0	2	1
11	10	9	10	9	11
9	10	11	9	99	11
10	9	9	11	10	10

F

1/9

1	1	1
1	1	1
1	1	1

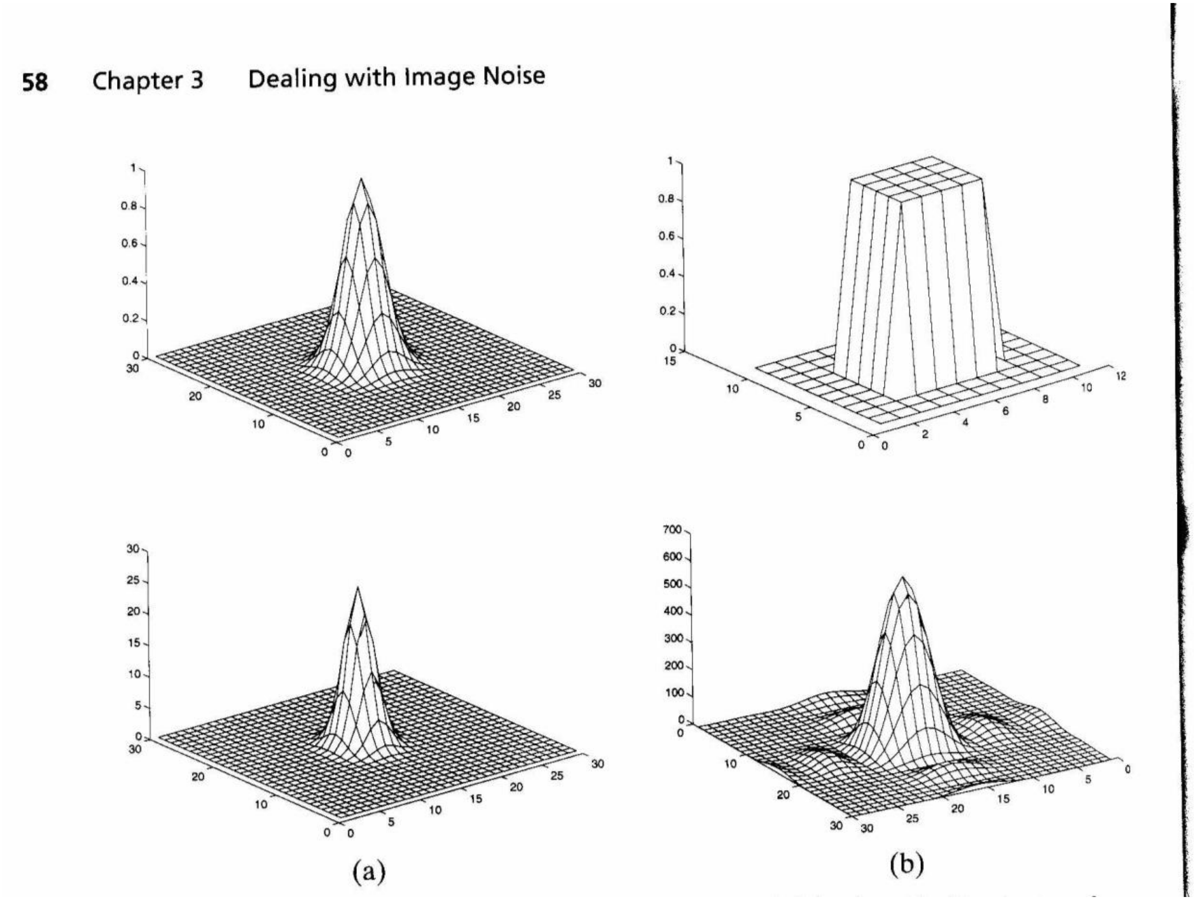
O

X	X	X	X	X	X
X	10				X
X					X
X					X
X					X
X	X	X	X	X	X

1/9.(10x1 + 11x1 + 10x1 + 9x1 + 10x1 + 11x1 + 10x1 + 9x1 + 10x1) =
1/9.(90) = 10

Examples

- Transform of box filter is sinc.
- Transform of Gaussian is Gaussian.



58 Chapter 3 Dealing with Image Noise

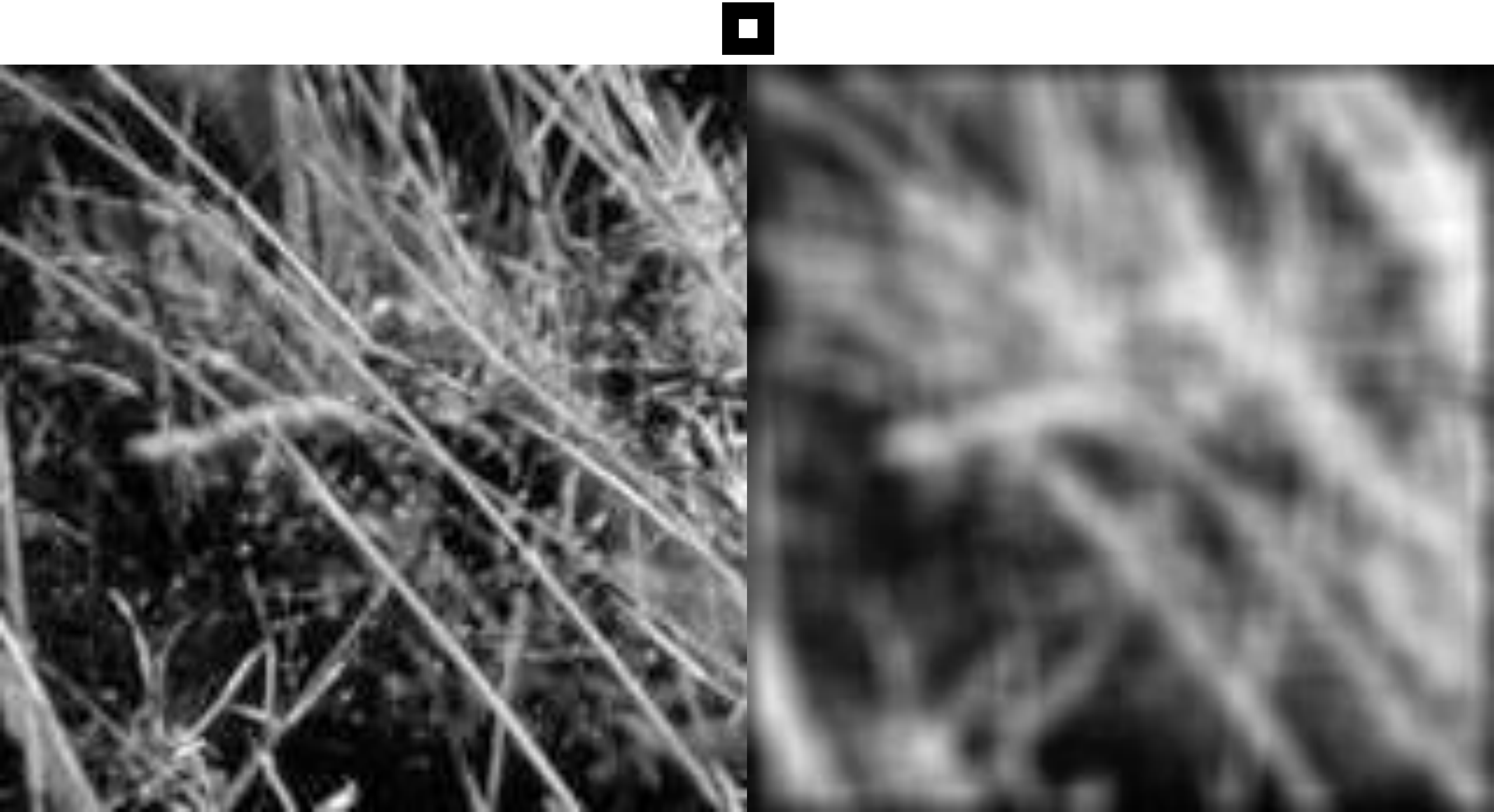
(a) (b)

(Trucco and Verri)

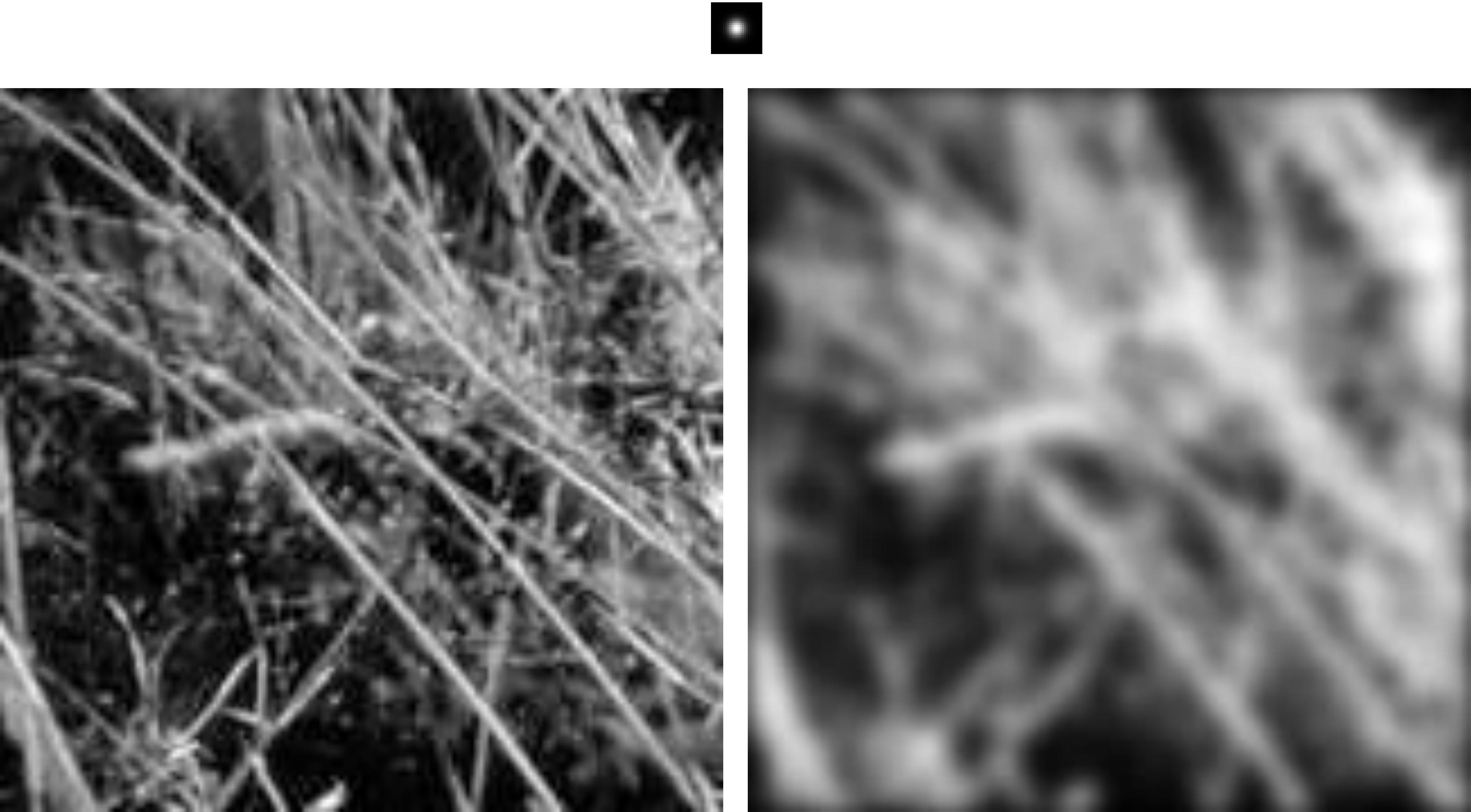
Implications

- Smoothing means removing high frequencies. This is one definition of scale.
- Sinc function explains artifacts.
- Need smoothing before subsampling to avoid aliasing.

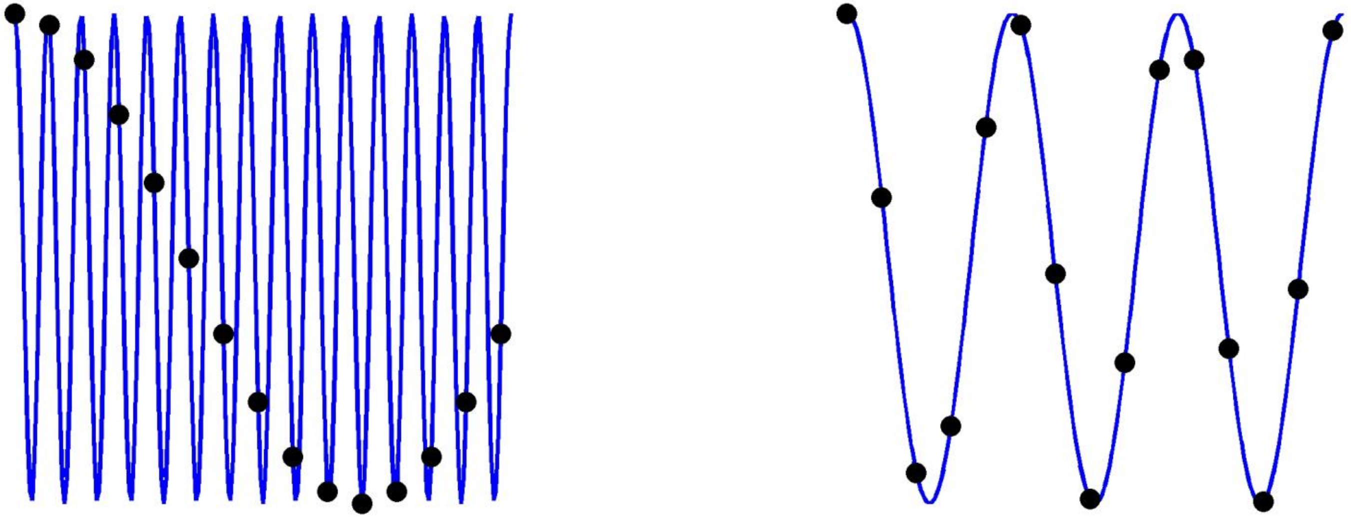
Example: Smoothing by Averaging



Smoothing with a Gaussian



Sampling



Sampling and the Nyquist rate

- **Aliasing** can arise when you sample a continuous signal or image
  - Demo applet http://www.cs.brown.edu/exploratories/freeSoftware/repository/edu/brown/cs/exploratories/applets/nyquist/nyquist_limit_java_plugin.html
  - occurs when your sampling rate is not high enough to capture the amount of detail in your image
  - formally, the image contains structure at different scales
    - called “frequencies” in the Fourier domain
  - the sampling rate must be high enough to capture the highest frequency in the image
- To avoid aliasing:
  - sampling rate > 2 * max frequency in the image
    - i.e., need more than two samples per period
  - This minimum sampling rate is called the **Nyquist rate**