

Projective geometry



[Ames Room](#)

Slides from Steve Seitz and Daniel DeMenthon

Projective geometry—what's it good for?

Uses of projective geometry

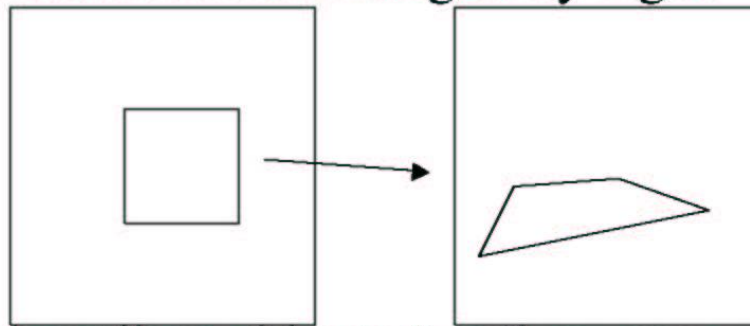
- Drawing
- Measurements
- Mathematics for projection
- Undistorting images
- Focus of expansion
- Camera pose estimation, match move
- Object recognition

Euclidean versus Projective Geometry

- Euclidean geometry describes shapes “as they are”

- Properties of objects that are unchanged by rigid motions

- » Lengths
- » Angles
- » Parallelism



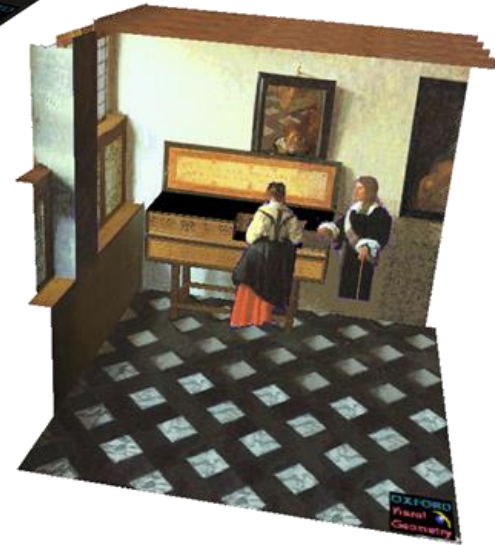
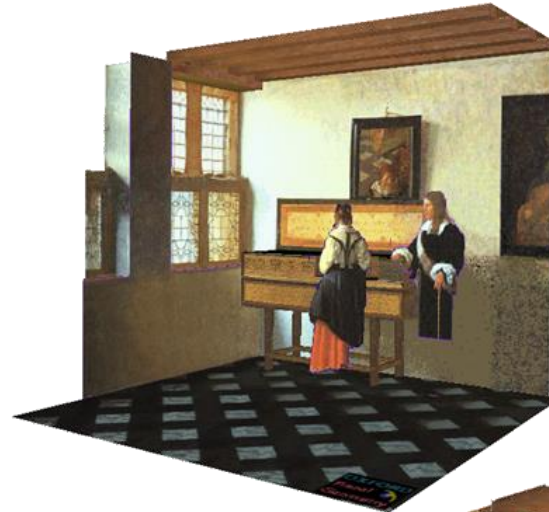
- Projective geometry describes objects “as they appear”

- Lengths, angles, parallelism become “distorted” when we look at objects
- Mathematical model for how images of the 3D world are formed.

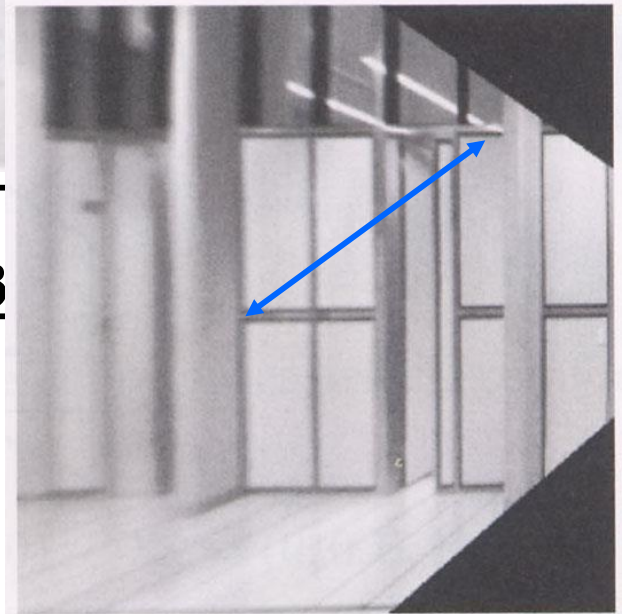
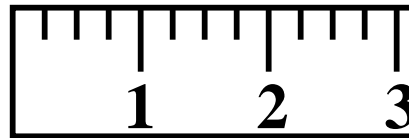
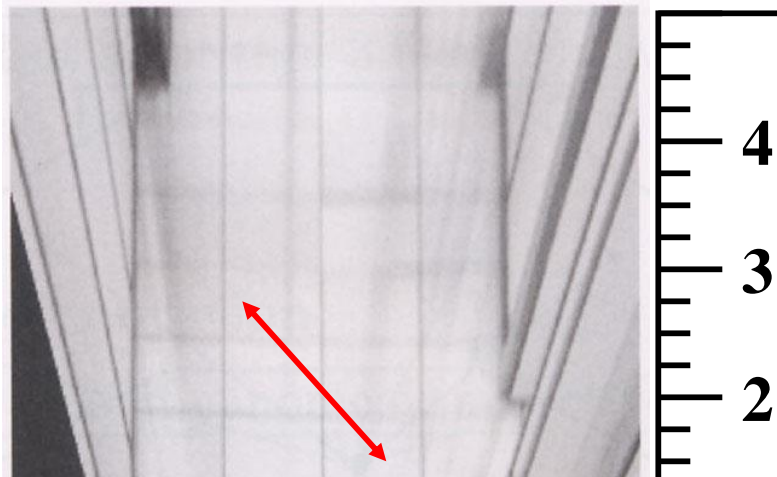
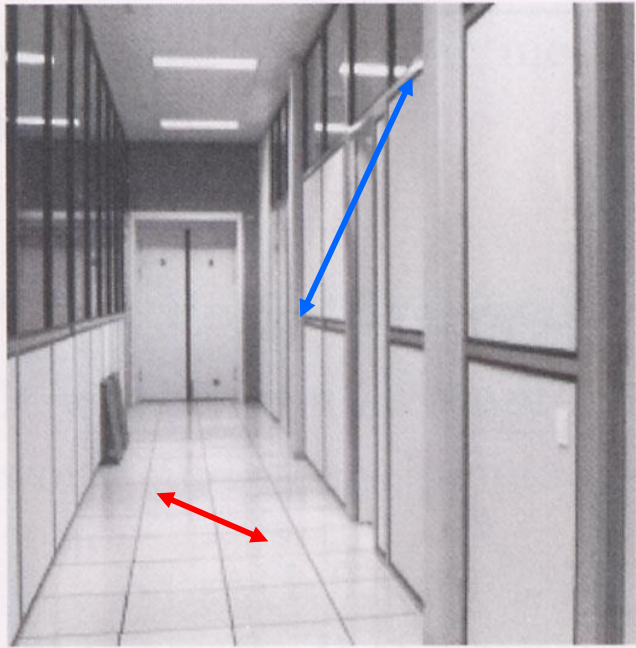
Applications of projective geometry



Vermeer's *Music Lesson*

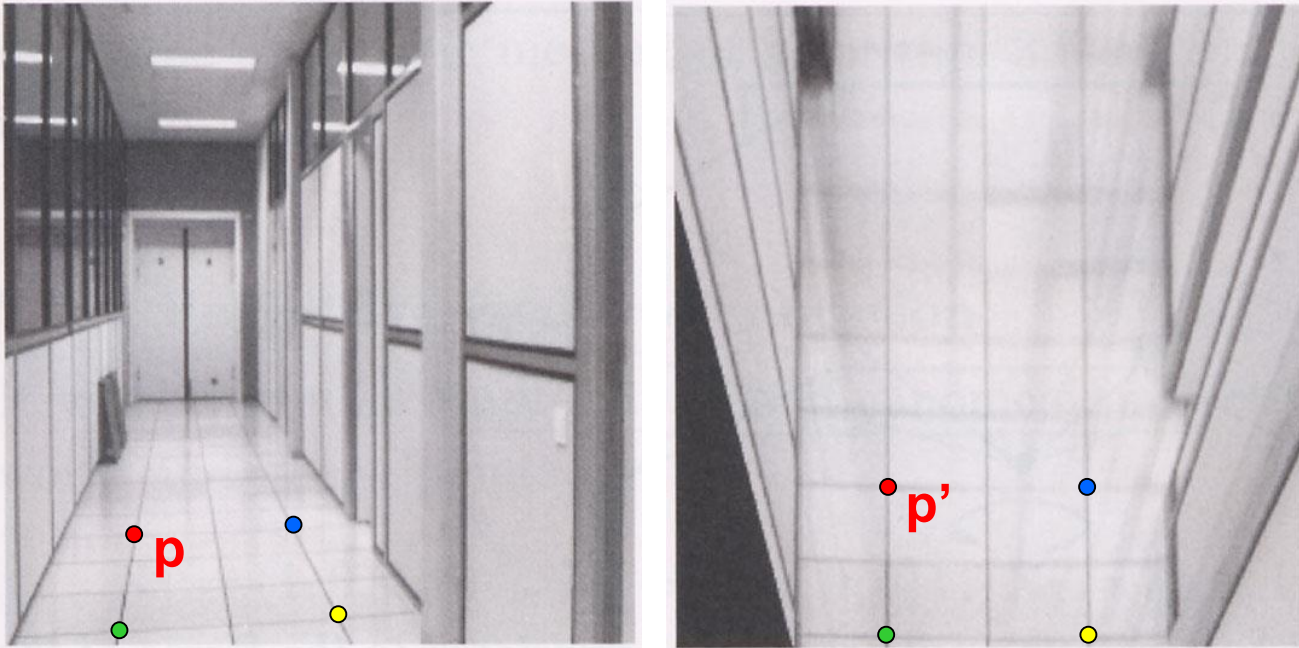


Measurements on planes



Approach: unwarp then measure
What kind of warp is this?

Image rectification



To unwarped (rectify) an image

- solve for homography \mathbf{H} given \mathbf{p} and \mathbf{p}'
- solve equations of the form: $w\mathbf{p}' = \mathbf{H}\mathbf{p}$
 - linear in unknowns: w and coefficients of \mathbf{H}
 - \mathbf{H} is defined up to an arbitrary scale factor
 - how many points are necessary to solve for \mathbf{H} ?

Solving for homographies

$$\begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$x'_i(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02}$$

$$y'_i(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12}$$

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -x'_i x_i & -x'_i y_i & -x'_i \\ 0 & 0 & 0 & x_i & y_i & 1 & -y'_i x_i & -y'_i y_i & -y'_i \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

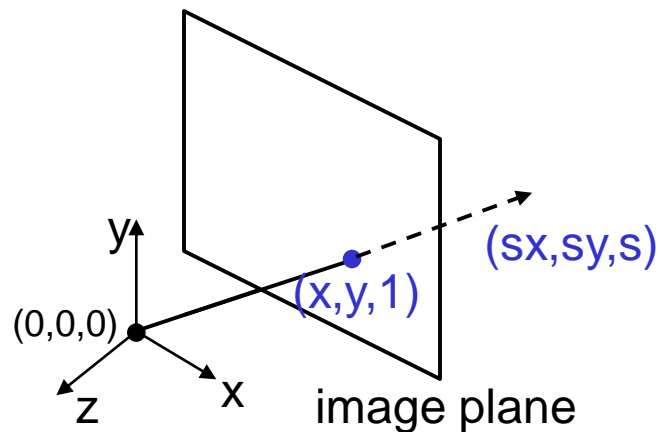
The projective plane

Why do we need homogeneous coordinates?

- represent points at infinity, homographies, perspective projection, multi-view relationships

What is the geometric intuition?

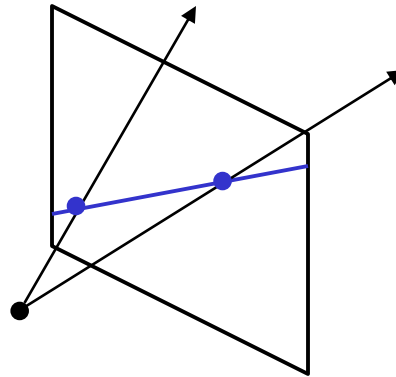
- a point in the image is a *ray* in projective space



- Each *point* (x,y) on the plane is represented by a *ray* (sx, sy, s)
 - all points on the ray are equivalent: $(x, y, 1) \cong (sx, sy, s)$

Projective lines

What does a line in the image correspond to in projective space?



- A line is a *plane* of rays through origin
 - all rays (x,y,z) satisfying: $ax + by + cz = 0$

in vector notation :

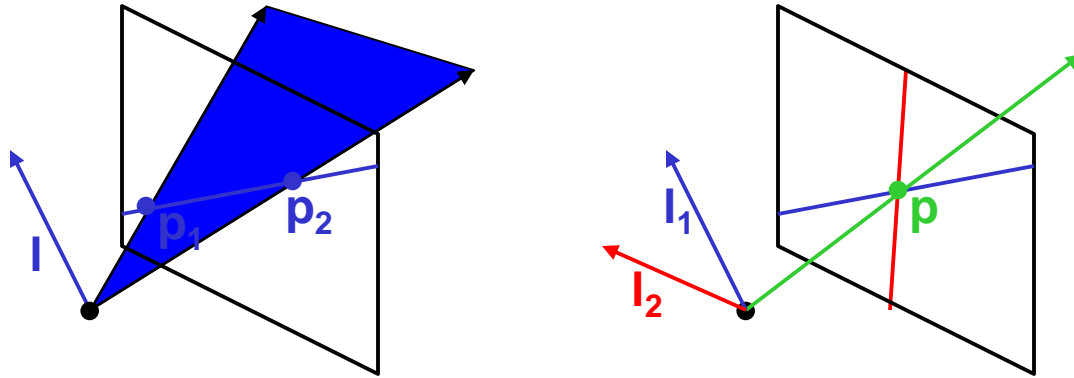
$$0 = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

l **p**

- A line is also represented as a homogeneous 3-vector **l**

Point and line duality

- A line \mathbf{l} is a homogeneous 3-vector
- It is \perp to every point (ray) \mathbf{p} on the line: $\mathbf{l} \mathbf{p} = 0$



What is the line \mathbf{l} spanned by rays \mathbf{p}_1 and \mathbf{p}_2 ?

- \mathbf{l} is \perp to \mathbf{p}_1 and $\mathbf{p}_2 \Rightarrow \mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$
- \mathbf{l} is the plane normal

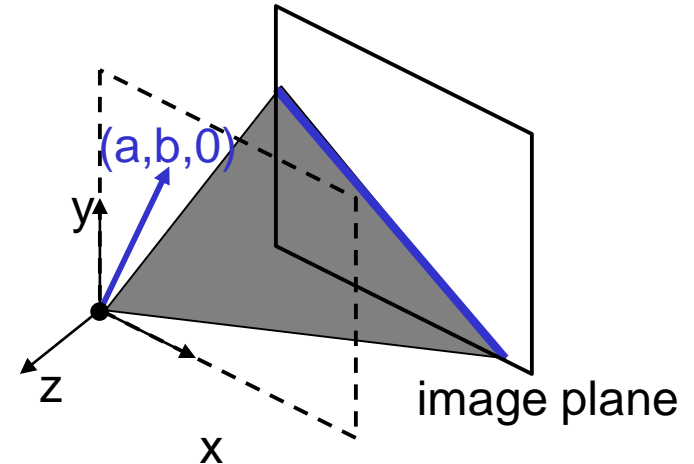
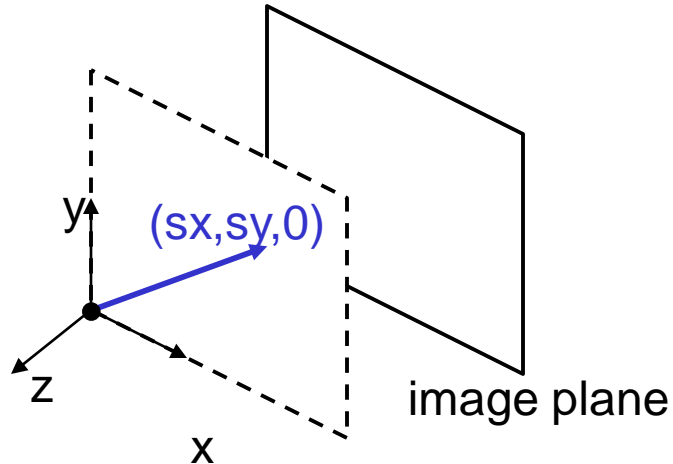
What is the intersection of two lines \mathbf{l}_1 and \mathbf{l}_2 ?

- \mathbf{p} is \perp to \mathbf{l}_1 and $\mathbf{l}_2 \Rightarrow \mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2$

Points and lines are *dual* in projective space

- given any formula, can switch the meanings of points and lines to get another formula

Ideal points and lines



Ideal point (“point at infinity”)

- $p \cong (x, y, 0)$ – parallel to image plane
- It has infinite image coordinates

Ideal line

- $l \cong (a, b, 0)$ – parallel to image plane
- Corresponds to a line in the image (finite coordinates)

Homographies of points and lines

Computed by 3x3 matrix multiplication

- To transform a point: $\mathbf{p}' = \mathbf{H}\mathbf{p}$
- To transform a line: $\mathbf{l}\mathbf{p}=0 \rightarrow \mathbf{l}'\mathbf{p}'=0$
 - $0 = \mathbf{l}\mathbf{p} = \mathbf{H}^{-1}\mathbf{H}\mathbf{l}\mathbf{p} = \mathbf{H}^{-1}\mathbf{l}'\mathbf{p}' \Rightarrow \mathbf{l}' = \mathbf{H}^{-1}\mathbf{l}$
 - lines are transformed by postmultiplication of \mathbf{H}^{-1}

Projective Transformations in a Plane

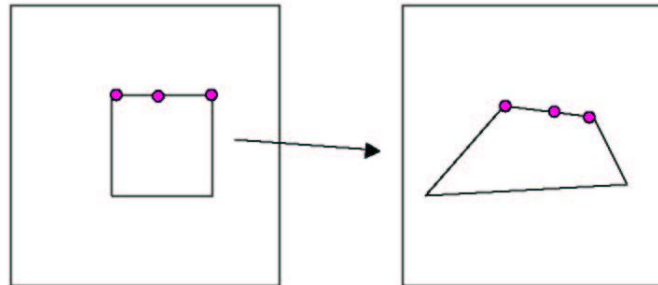


- ***Projectivity***

- Mapping from points in plane to points in plane
- 3 aligned points are mapped to 3 aligned points


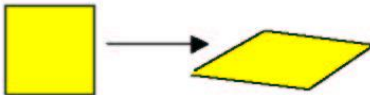

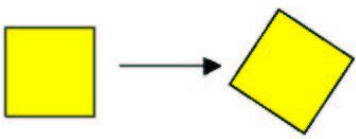
- Also called

- ***Collineation***
- ***Homography***



Special Projectivities



		Invariants	
Projectivity 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	Collinearity, Cross-ratios	
Affine transform 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_x \\ 0 & 0 & 1 \end{bmatrix}$	Parallelism, Ratios of areas, Length ratios	
Similarity 4 dof	$\begin{bmatrix} s r_{11} & s r_{12} & t_x \\ s r_{21} & s r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	Angles, Length ratios	
Euclidean transform 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	Angles, Lengths, Areas	

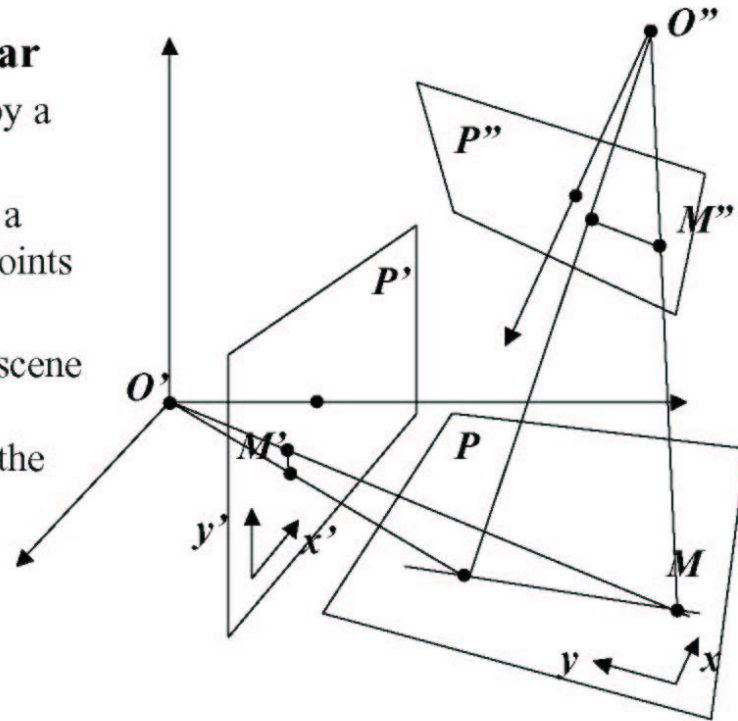
Examples of Projective Transformations



- Central projection maps **planar scene** points to image plane by a projectivity
 - True because all points on a scene line are mapped to points on its image line
- The image of the same planar scene from a second camera can be obtained from the image from the first camera by a projectivity
 - True because

$$\mathbf{x}'_i = \mathbf{H}' \mathbf{x}_i, \mathbf{x}''_i = \mathbf{H}'' \mathbf{x}_i$$

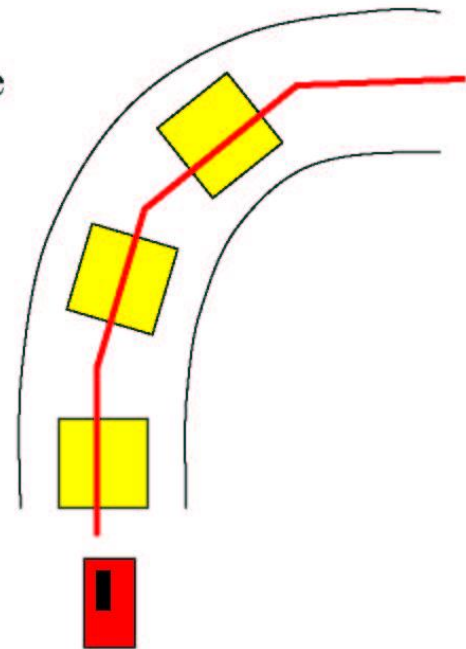
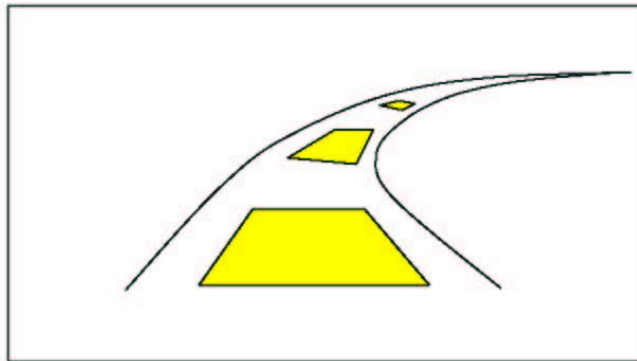
$$\text{so } \mathbf{x}''_i = \mathbf{H}'' \mathbf{H}'^{-1} \mathbf{x}'_i$$



Example of Application



- Robot going down the road
- Large squares painted on the road to make it easier
- Find road shape without perspective distortion from image
 - Use corners of squares: coordinates of 4 points allow us to compute matrix \mathbf{H}
 - Then use matrix \mathbf{H} to compute 3D road shape



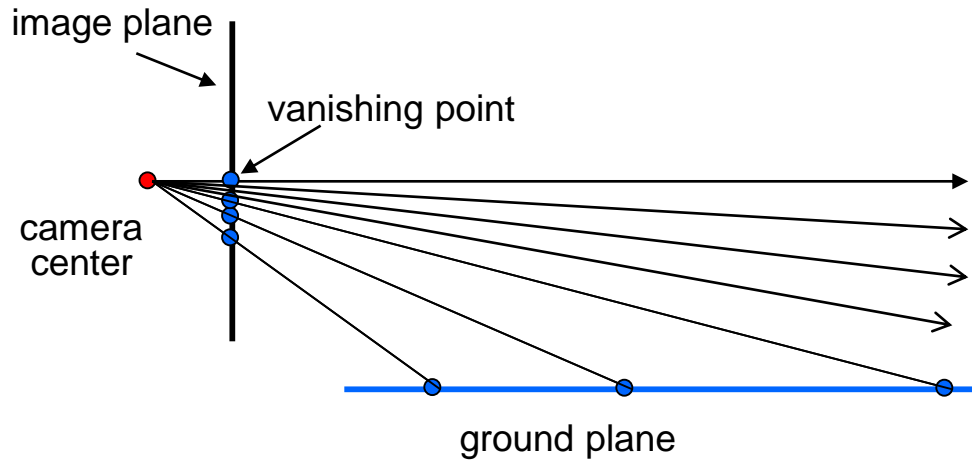
3D to 2D: “perspective” projection

Matrix Projection:
$$\mathbf{p} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \mathbf{PP}$$

What is *not* preserved under perspective projection?

What IS preserved?

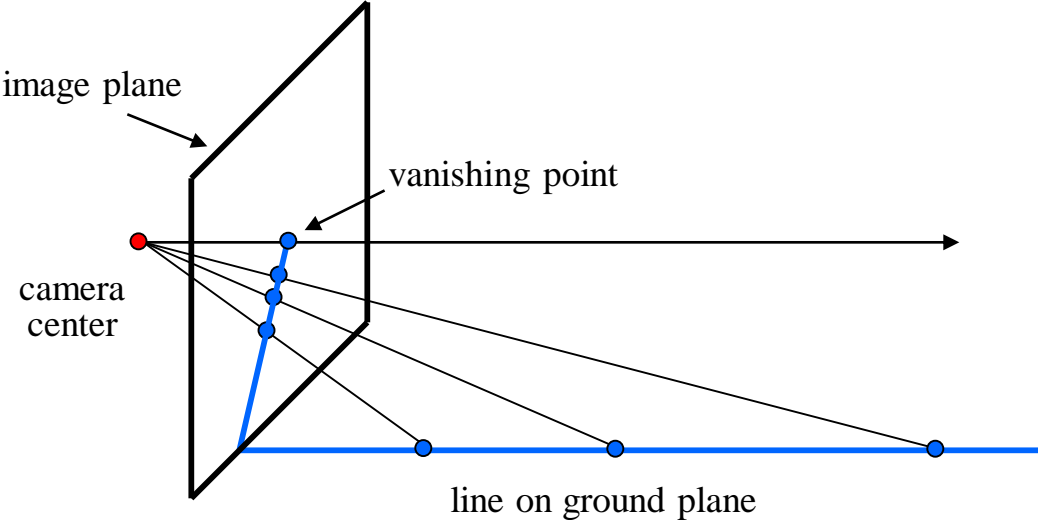
Vanishing points



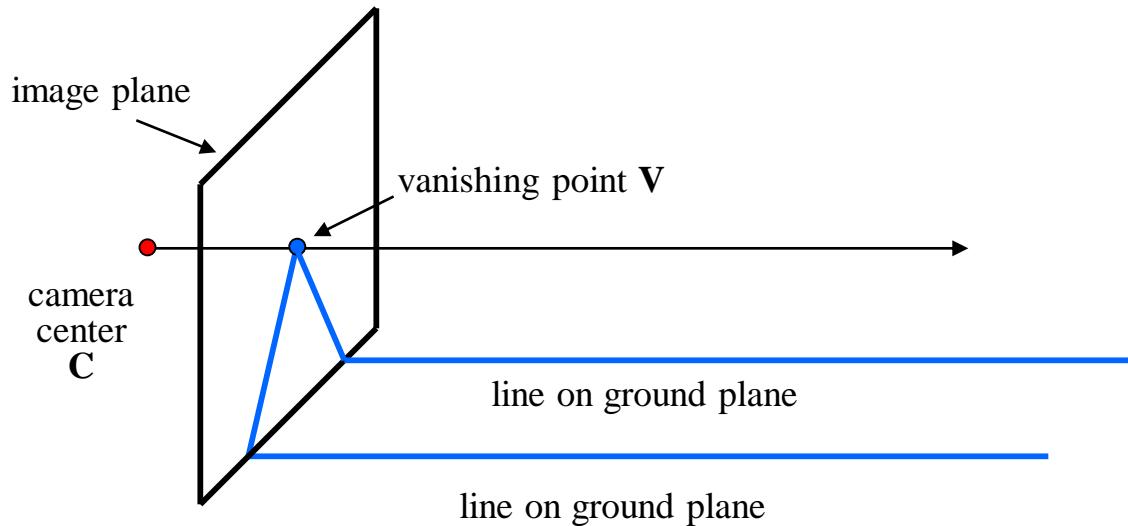
Vanishing point

- projection of a point at infinity

Vanishing points (2D)



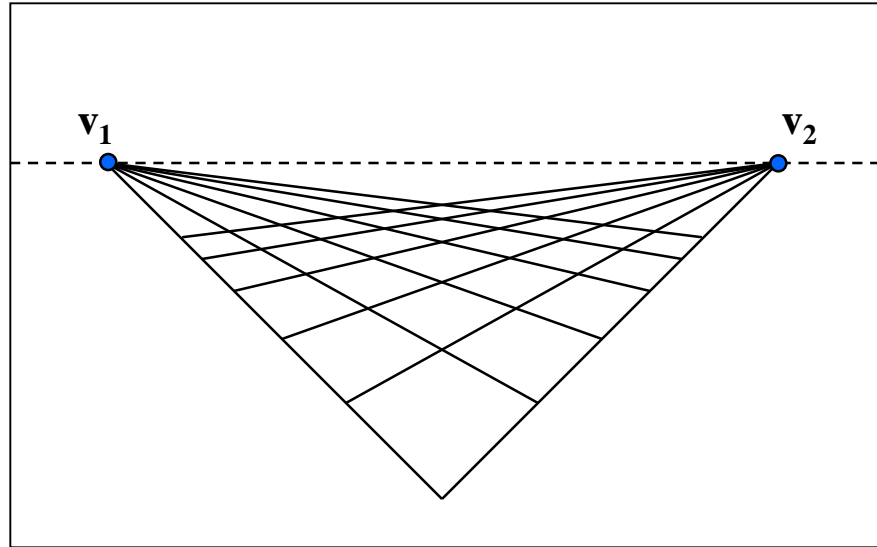
Vanishing points



Properties

- Any two parallel lines have the same vanishing point v
- The ray from C through v is parallel to the lines
- An image may have more than one vanishing point

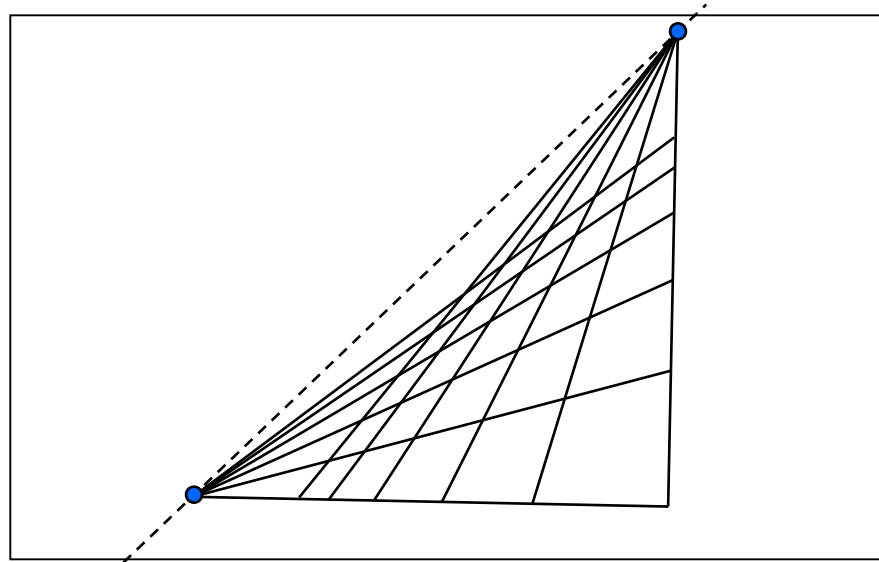
Vanishing lines



Multiple Vanishing Points

- Any set of parallel lines on the plane define a vanishing point
- The union of all of these vanishing points is the *horizon line*
– also called *vanishing line*
- Note that different planes define different vanishing lines

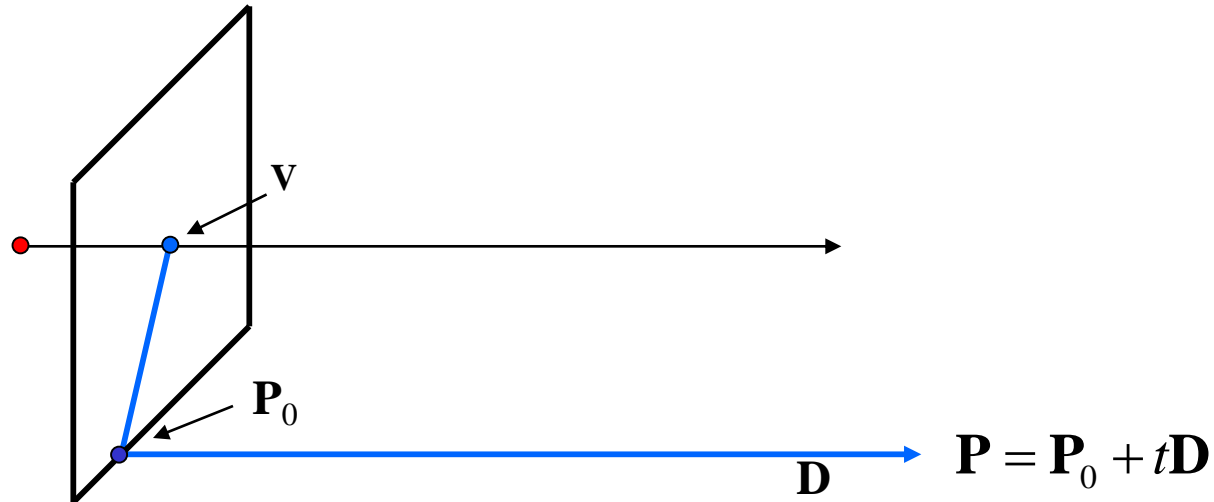
Vanishing lines



Multiple Vanishing Points

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Computing vanishing points

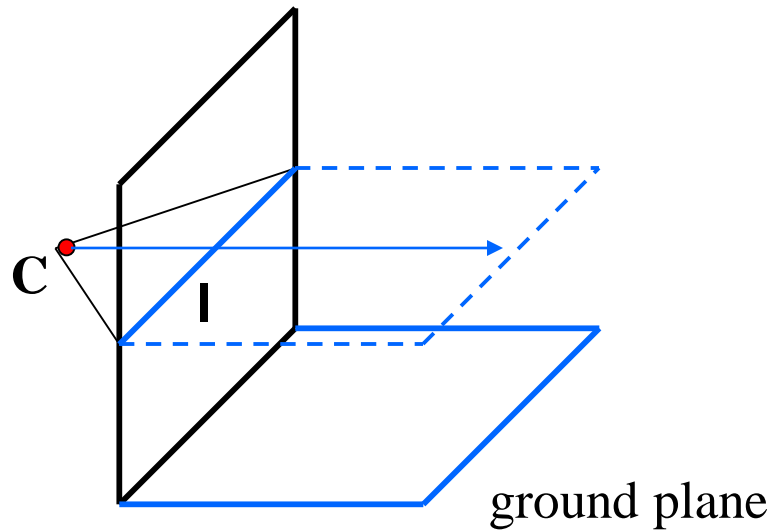


$$\mathbf{P}_t = \begin{bmatrix} P_X + tD_X \\ P_Y + tD_Y \\ P_Z + tD_Z \\ 1 \end{bmatrix} \cong \begin{bmatrix} P_X / t + D_X \\ P_Y / t + D_Y \\ P_Z / t + D_Z \\ 1/t \end{bmatrix} \quad t \rightarrow \infty \quad \mathbf{P}_\infty \cong \begin{bmatrix} D_X \\ D_Y \\ D_Z \\ 0 \end{bmatrix}$$

Properties $\mathbf{v} = \mathbf{IIP}_\infty$

- \mathbf{P}_∞ is a point at *infinity*, \mathbf{v} is its projection
- They depend only on line *direction*
- Parallel lines $\mathbf{P}_0 + t\mathbf{D}$, $\mathbf{P}_1 + t\mathbf{D}$ intersect at \mathbf{P}_∞

Computing vanishing lines

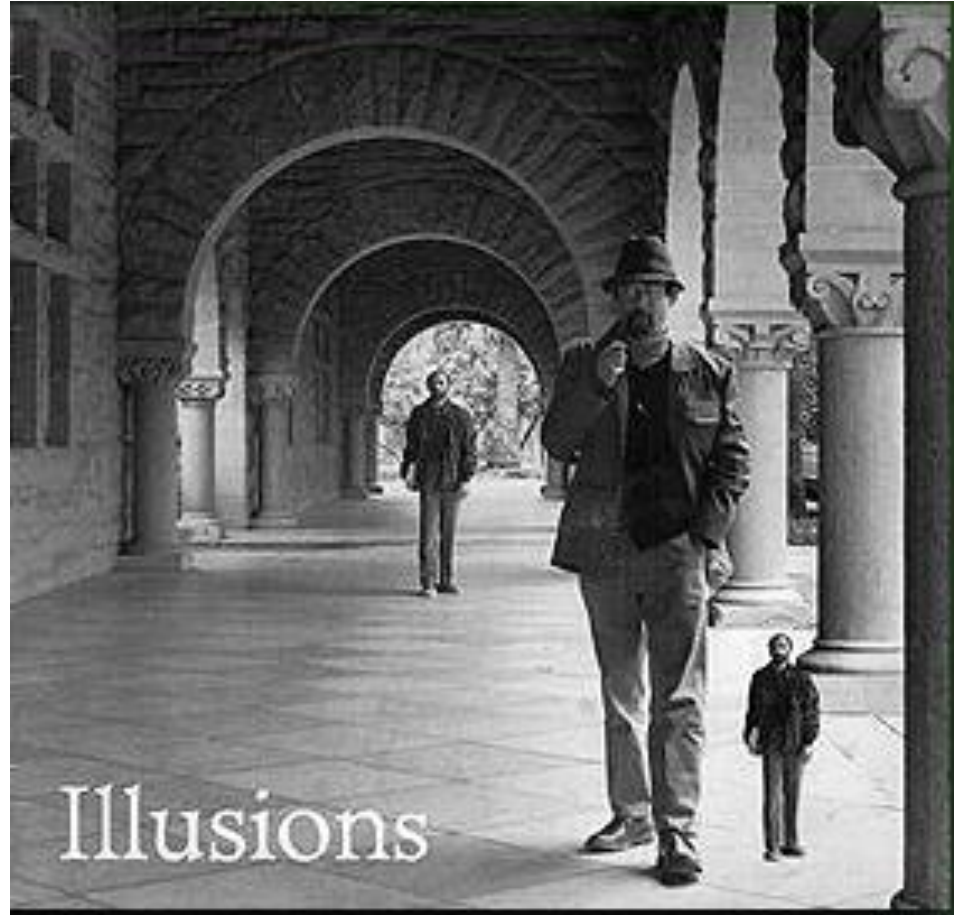
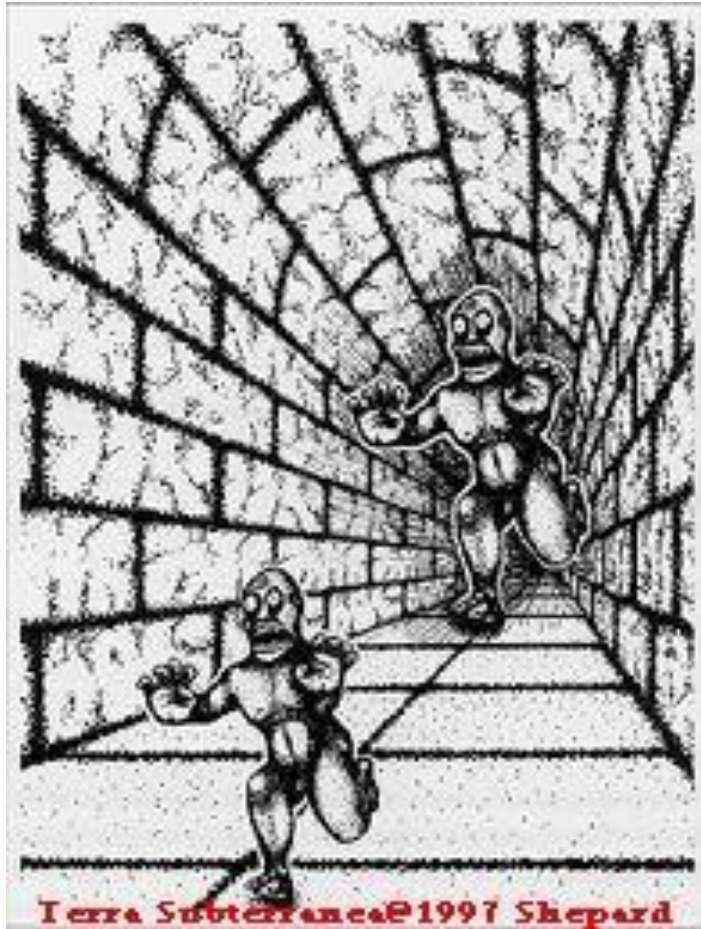


Properties

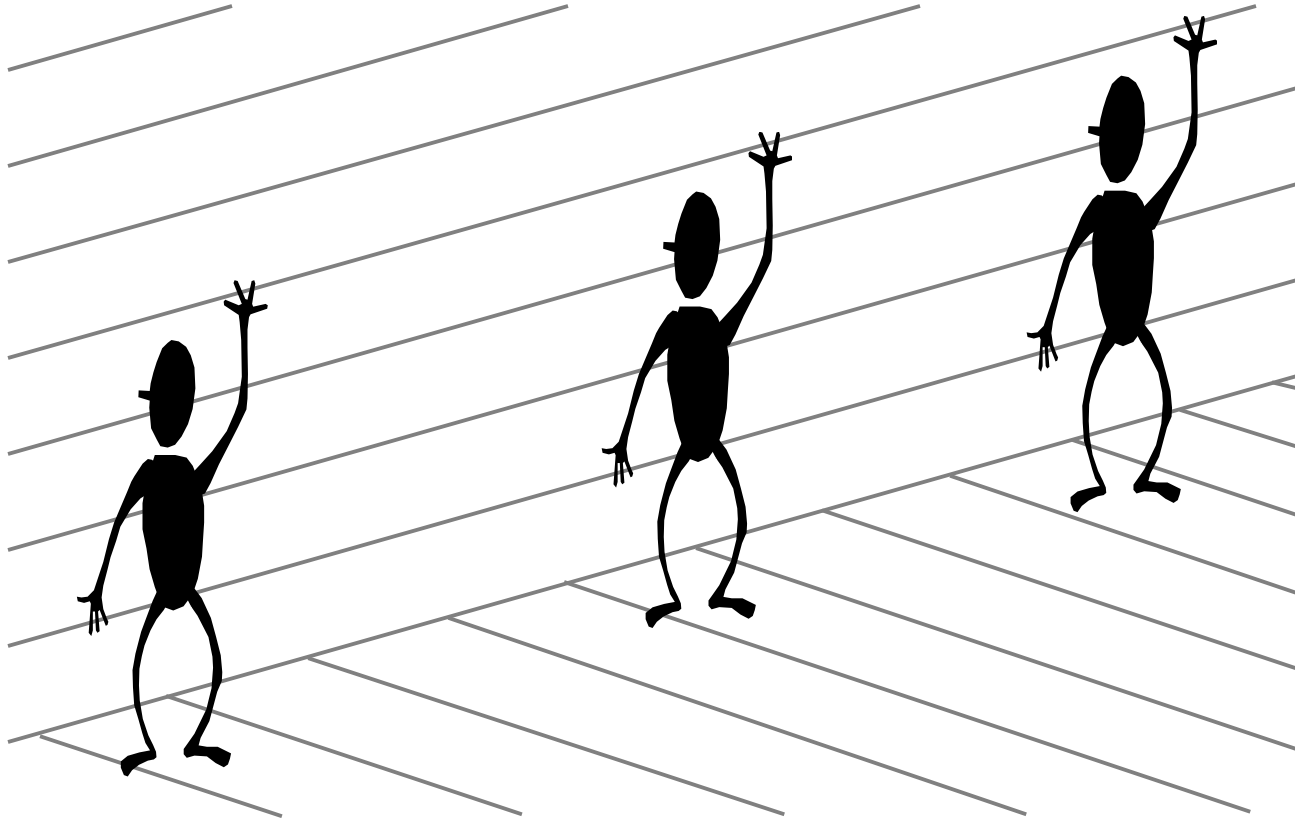
- I is intersection of horizontal plane through C with image plane
- Compute I from two sets of parallel lines on ground plane
- All points at same height as C project to I
 - points higher than C project above I
- Provides way of comparing height of objects in the scene



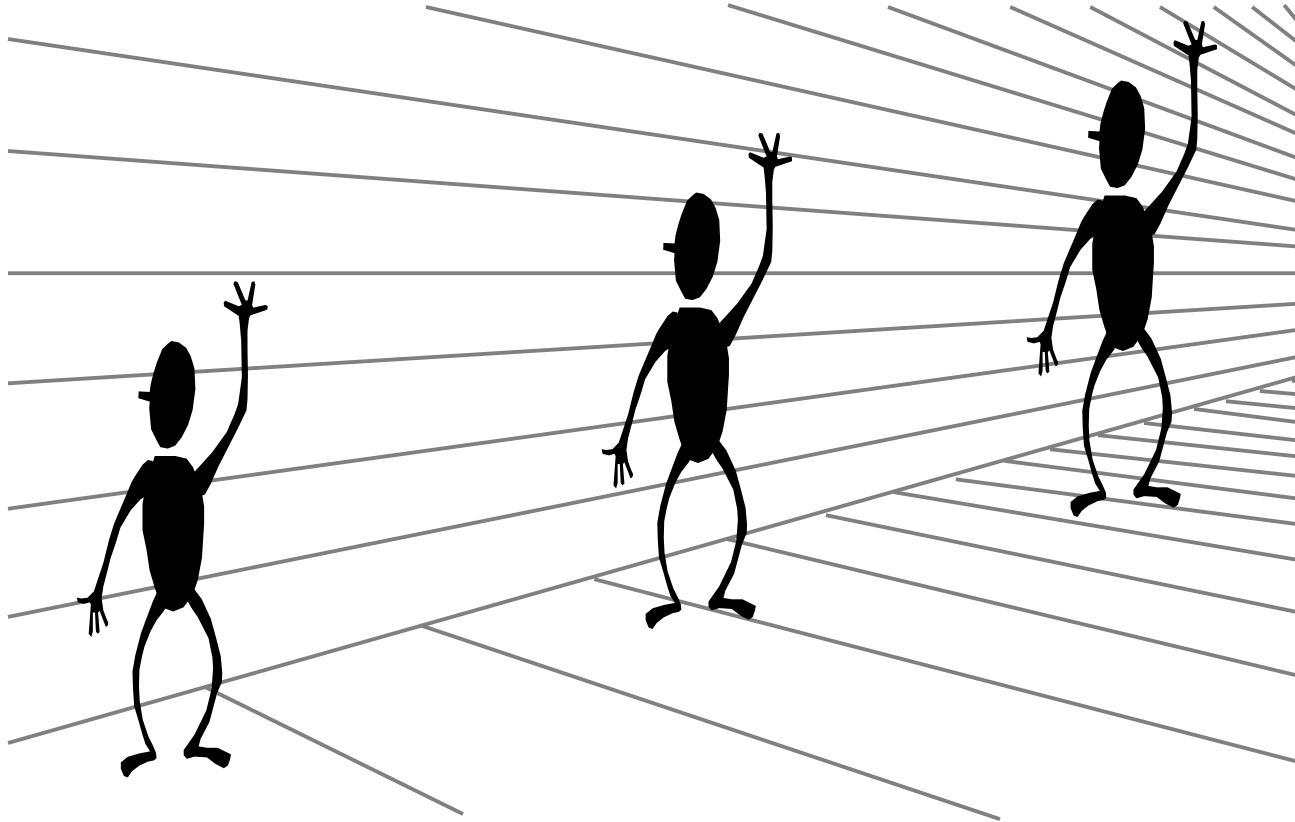
Fun with vanishing points



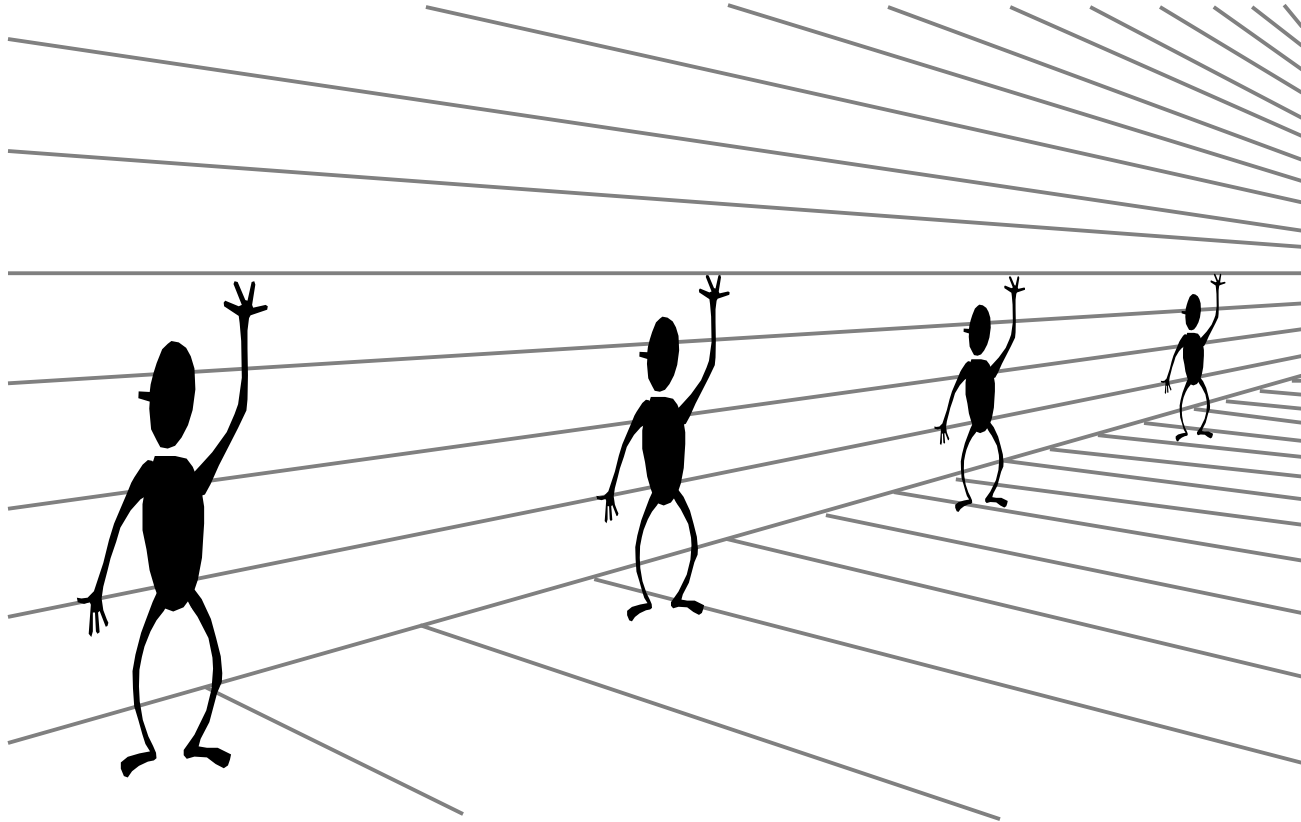
Perspective cues



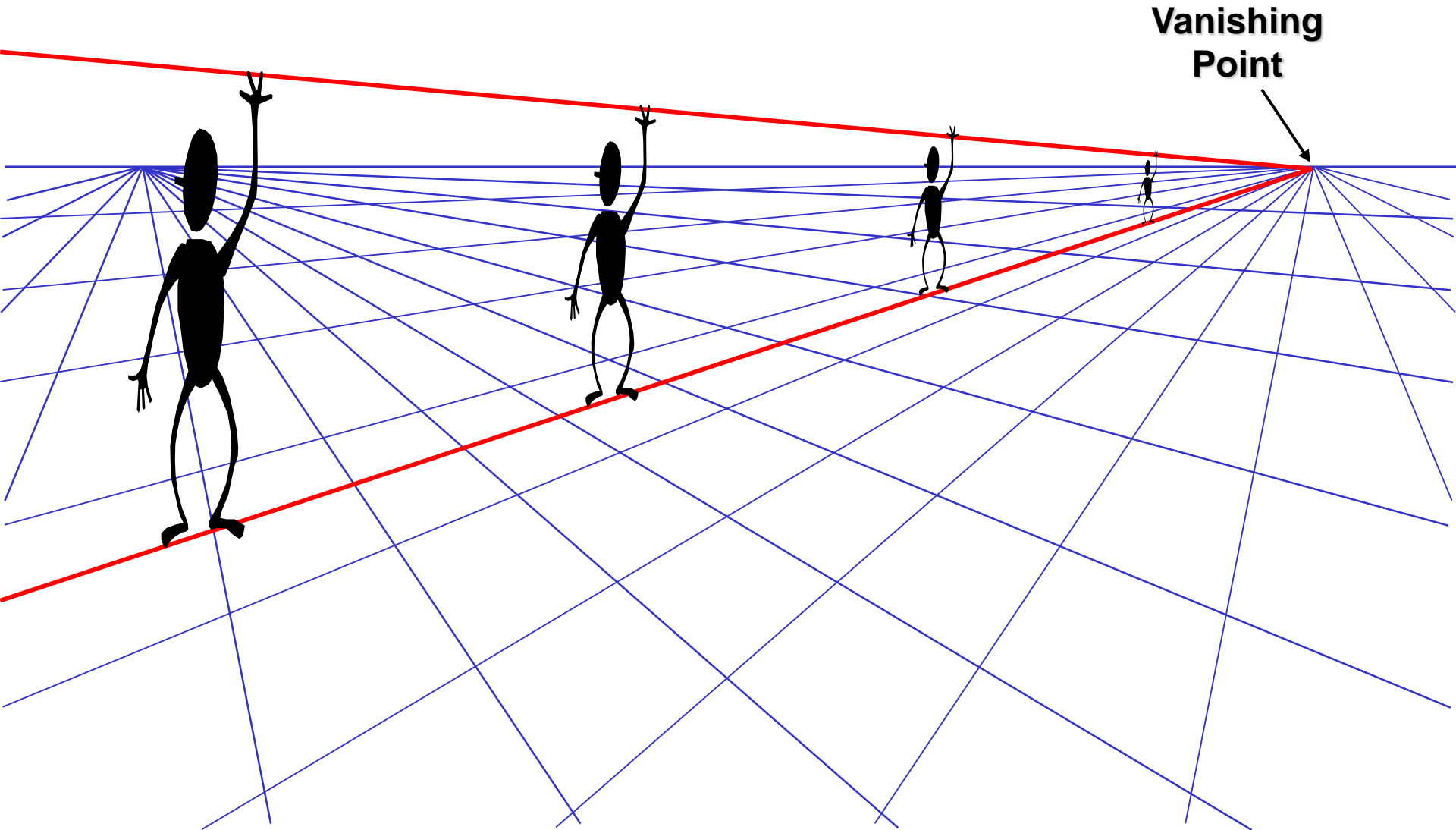
Perspective cues



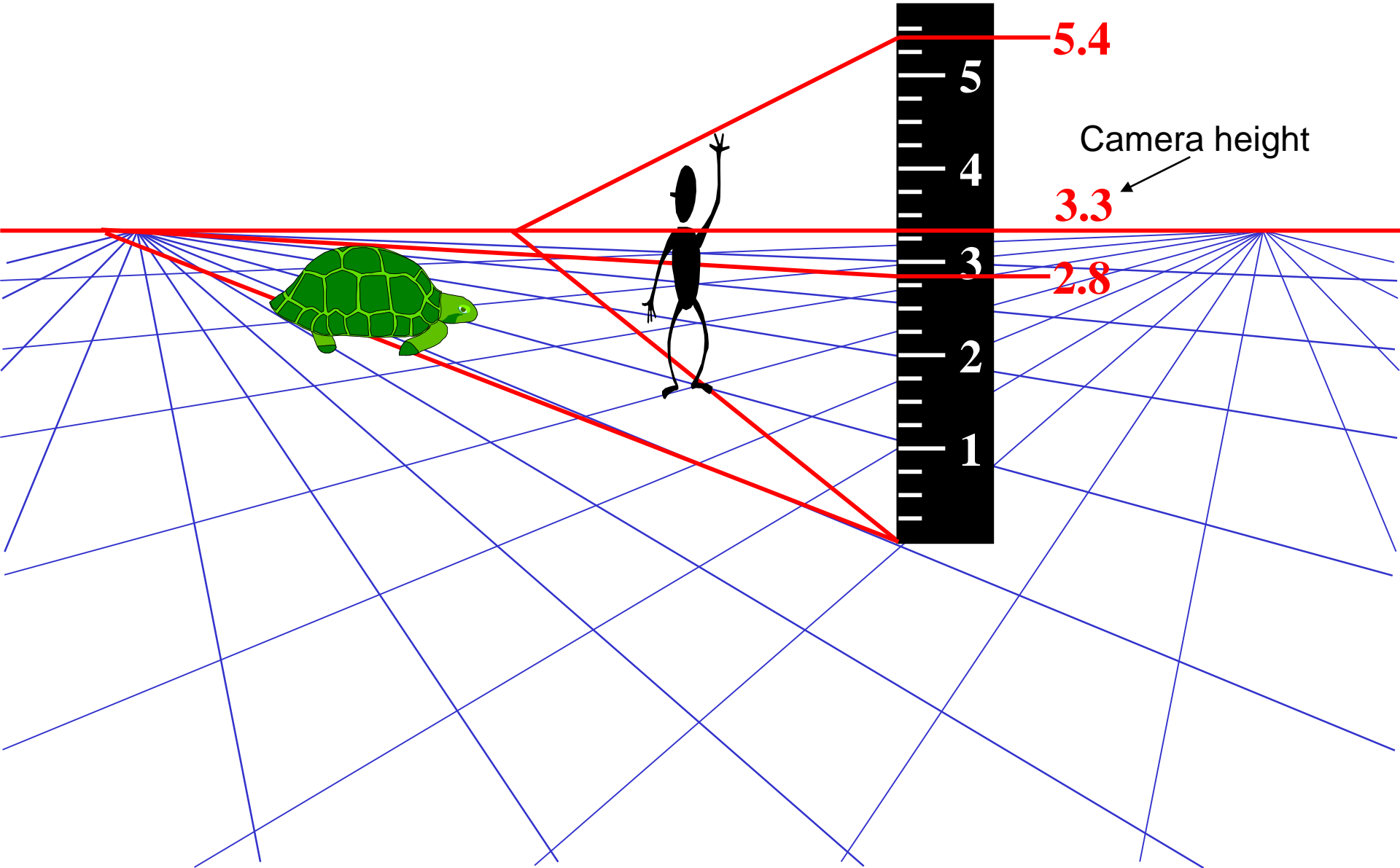
Perspective cues



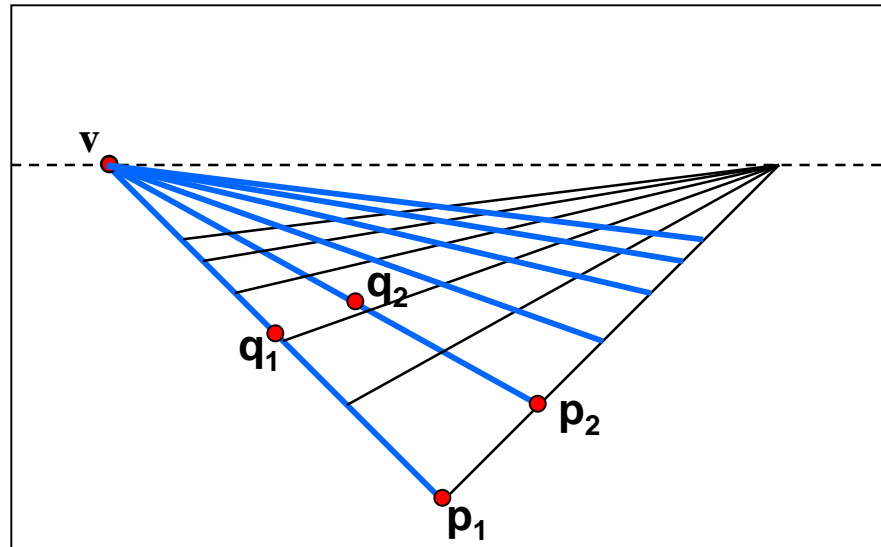
Comparing heights



Measuring height



Computing vanishing points (from lines)



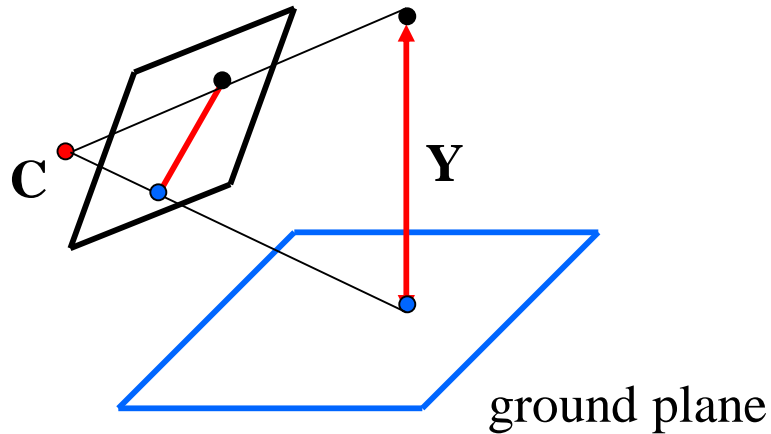
Intersect p_1q_1 with p_2q_2

$$v = (p_1 \times q_1) \times (p_2 \times q_2)$$

Least squares version

- Better to use more than two lines and compute the “closest” point of intersection

Measuring height without a ruler



Compute Y from image measurements

- Need more than vanishing points to do this

Cross-Ratio Invariance in 1D



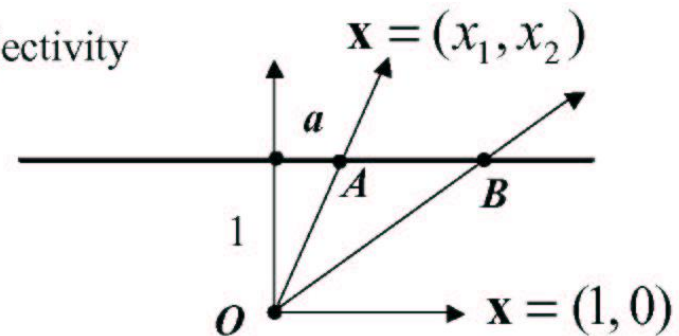
- Cross-ratio of 4 points A, B, C, D on a line is defined as

$$\text{Cross}(A, B, C, D) = \frac{|AB|}{|AD|} \div \frac{|CB|}{|CD|} \text{ with } |AB| = \det \begin{bmatrix} x_{A1} & x_{B1} \\ x_{A2} & x_{B2} \end{bmatrix}$$

- Cross-ratio is not dependent on which particular homogeneous representation of the points is selected: scales cancel between numerator and denominator. For $A = (a, 1)$, $B = (b, 1)$, etc, we get

$$\text{Cross}(A, B, C, D) = \frac{a-b}{a-d} \div \frac{c-b}{c-d}$$

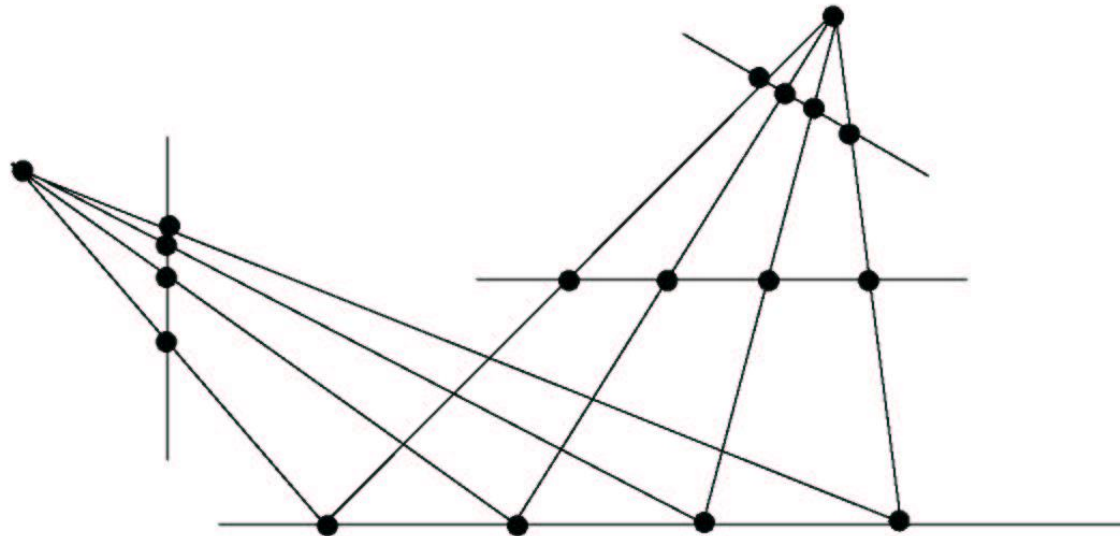
- Cross-ratio is invariant under any projectivity



Cross-Ratio Invariance in 1D



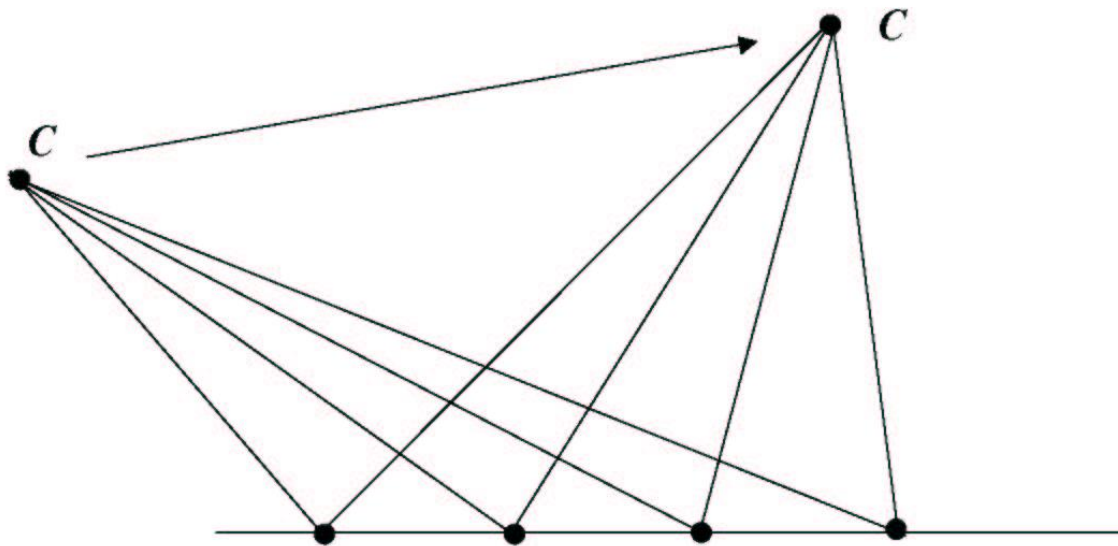
- For the 4 sets of collinear points in the figure, the cross-ratio for corresponding points has the same value



Cross-Ratio Invariance between Lines



- The cross-ratio between 4 lines forming a *pencil* is invariant when the point of intersection C is moved
- It is equal to the cross-ratio of the 4 points

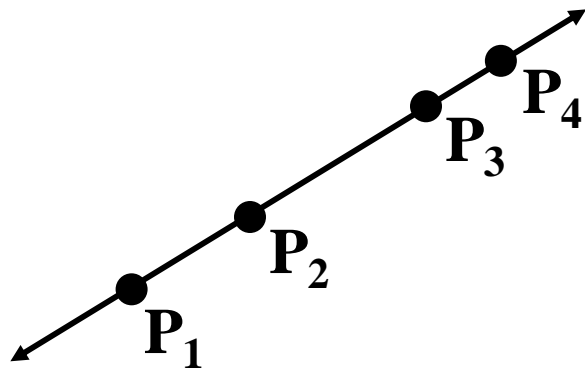


The cross ratio

A Projective Invariant

- Something that does not change under projective transformations (including perspective projection)

The cross-ratio of 4 collinear points



$$\frac{\| \mathbf{P}_3 - \mathbf{P}_1 \| \| \mathbf{P}_4 - \mathbf{P}_2 \|}{\| \mathbf{P}_3 - \mathbf{P}_2 \| \| \mathbf{P}_4 - \mathbf{P}_1 \|}$$

$$\mathbf{P}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

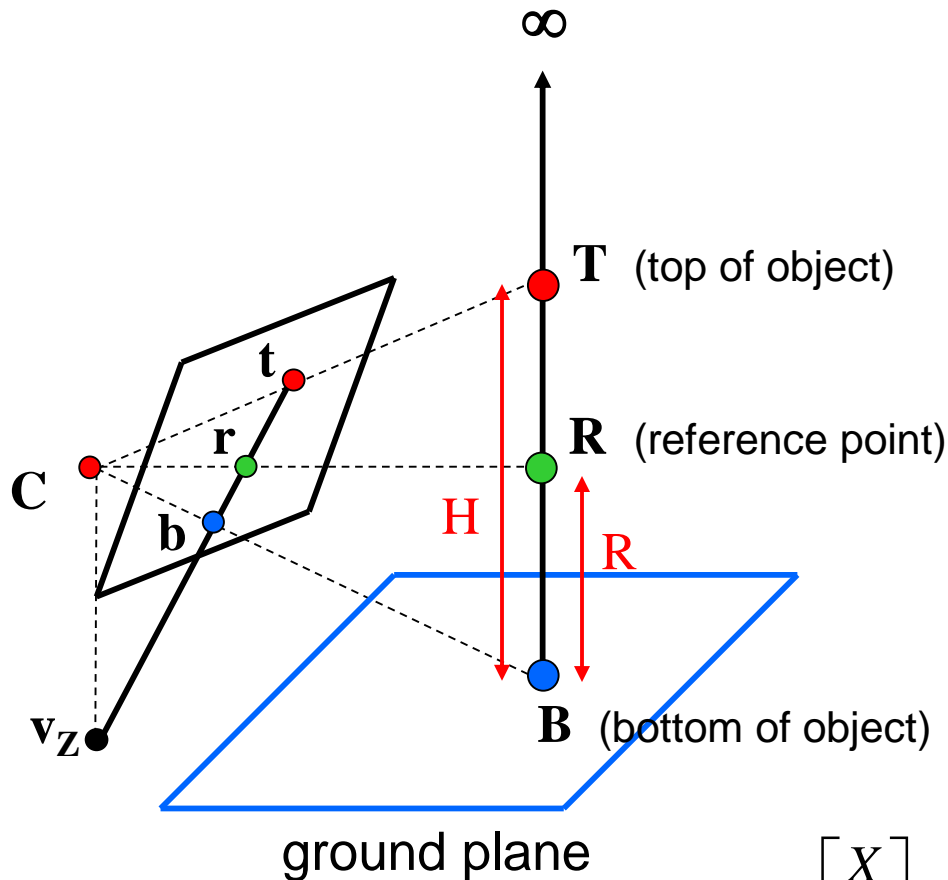
Can permute the point ordering

$$\frac{\| \mathbf{P}_1 - \mathbf{P}_3 \| \| \mathbf{P}_4 - \mathbf{P}_2 \|}{\| \mathbf{P}_1 - \mathbf{P}_2 \| \| \mathbf{P}_4 - \mathbf{P}_3 \|}$$

- $4! = 24$ different orders (but only 6 distinct values)

This is the fundamental invariant of projective geometry

Measuring height



scene points represented as $\mathbf{P} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$

image points as $\mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

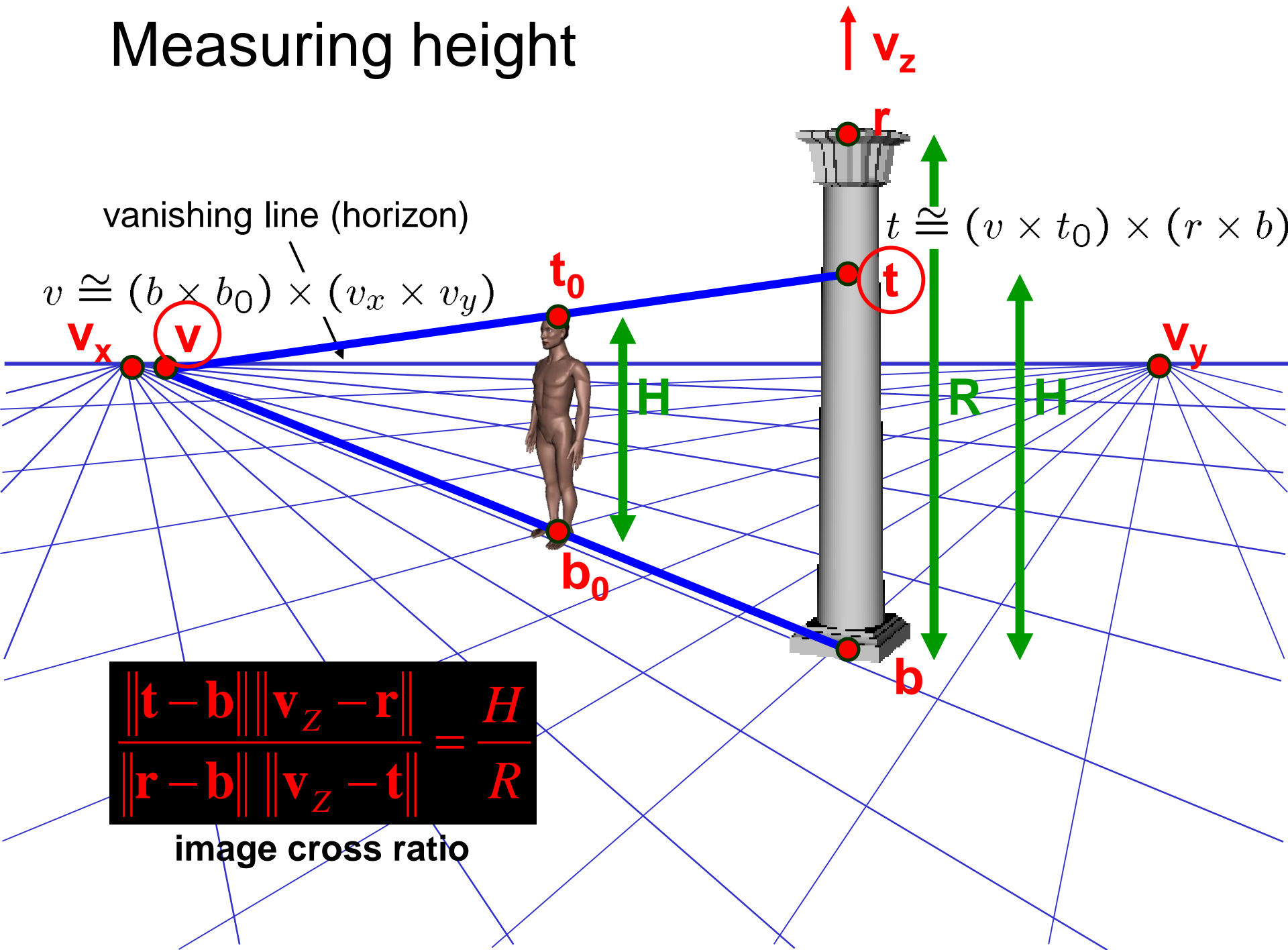
$$\frac{\|\mathbf{T} - \mathbf{B}\| \|\infty - \mathbf{R}\|}{\|\mathbf{R} - \mathbf{B}\| \|\infty - \mathbf{T}\|} = \frac{H}{R}$$

scene cross ratio

$$\frac{\|\mathbf{t} - \mathbf{b}\| \|v_z - \mathbf{r}\|}{\|\mathbf{r} - \mathbf{b}\| \|v_z - \mathbf{t}\|} = \frac{H}{R}$$

image cross ratio

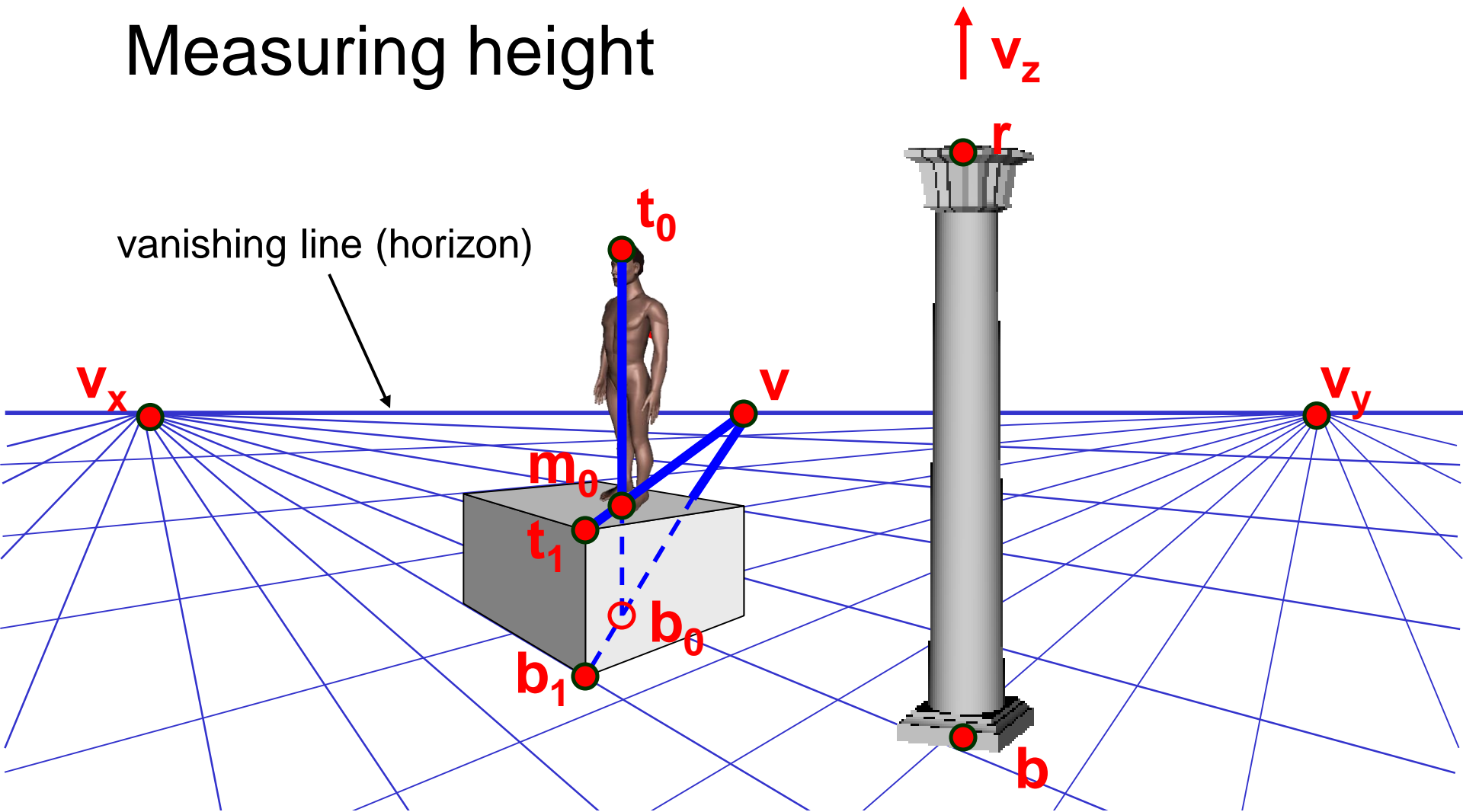
Measuring height



$$\frac{\|t - b\| \|v_z - r\|}{\|r - b\| \|v_z - t\|} = \frac{H}{R}$$

image cross ratio

Measuring height



What if the point on the ground plane b_0 is not known?

- Here the guy is standing on the box, height of box is known
- Use one side of the box to help find b_0 as shown above

3D projective geometry

These concepts generalize naturally to 3D

- Homogeneous coordinates
 - Projective 3D points have four coords: $\mathbf{P} = (X, Y, Z, W)$
- Duality
 - A plane \mathbf{N} is also represented by a 4-vector
 - Points and planes are dual in 3D: $\mathbf{N} \mathbf{P} = 0$
- Projective transformations
 - Represented by 4x4 matrices \mathbf{T} : $\mathbf{P}' = \mathbf{T}\mathbf{P}$, $\mathbf{N}' = \mathbf{N} \mathbf{T}^{-1}$

Projective Geometry in 3D

- Space P_3 is called projective space
- A point in 3D space is defined by 4 numbers (x_1, x_2, x_3, x_4)
- A plane is also defined by 4 numbers (u_1, u_2, u_3, u_4)
- Equation of plane is
$$\sum_{i=1}^4 u_i x_i = 0$$
- The plane at infinity is the plane $(0,0,0,1)$. Its equation is $x_4=0$
- The points $(x_1, x_2, x_3, 0)$ belong to that plane in the direction (x_1, x_2, x_3) of Euclidean space
- A line is defined as the set of points that are a linear combination of two points P_1 and P_2
- The cross-ratio of 4 planes is equal to the cross-ratio of the lines of intersection with a fifth plane

