ASSIGNMENT 1 CMSC 858K (Fall 2015)

Due in class on Thursday, September 17.

1. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.
   (a) [2 points] $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$
   (b) [2 points] $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$
   (c) [2 points] $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

2. *Unitary operations and measurements.* Consider the state
$$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle.$$
   (a) [2 points] Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.
   (b) [3 points] Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?
   (c) [3 points] Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?
   (d) [2 points] Suppose $|\phi\rangle$ is measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

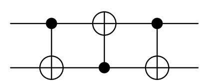
3. *Distinguishing quantum states.* [6 points] Let θ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state
$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle$$
(but does not tell you which). Describe a procedure for guessing which state you were given, succeeding with as high a probability as possible. Express your procedure as a unitary operation followed by a measurement in the computational basis. Also indicate the success probability of your procedure. (You do not need to prove that your procedure is optimal, but your grade will depend on how close your procedure is to optimal.)

4. *The Hadamard gate and qubit rotations*
   (a) [3 points] Suppose that $(n_x, n_y, n_z) \in \mathbb{R}^3$ is a unit vector and $\theta \in \mathbb{R}$. Show that
$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i\sin(\tfrac{\theta}{2})\,(n_x X + n_y Y + n_z Z).$$
   (b) [2 points] Find a unit vector $(n_x, n_y, n_z) \in \mathbb{R}^3$ and numbers $\phi, \theta \in \mathbb{R}$ so that
$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$
   where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

(c) [3 points] Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathbb{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

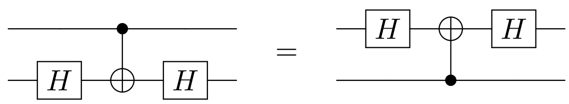
5. *Circuit identities.*

   (a) [2 points] What does the following circuit do? Show that your answer is correct.

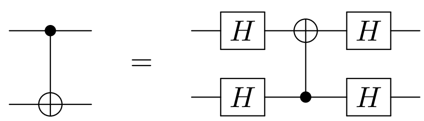
   

   (b) [1 point] Verify that $HXH = Z$, where H is the Hadamard gate and X, Z denote Pauli matrices.

   (c) [3 points] Verify the following circuit identity:

   (d) [2 points] Verify the following circuit identity:

   Give an interpretation of this identity.

6. *Teleporting through a Hadamard gate.*

   (a) [1 point] Write the state $(I \otimes H)|\beta_{00}\rangle$ in the computational basis.

   (b) [3 points] Suppose Alice has a qubit in the state $|\psi\rangle$ and also, Alice and Bob share a copy of the state $(I \otimes H)|\beta_{00}\rangle$. If Alice measures her two qubits in the Bell basis, what are the probabilities of the four possible outcomes, and in each case, what is the post-measurement state for Bob?

   (c) [2 points] Suppose Alice sends her measurement result to Bob. In each possible case, what operation should Bob perform in order to have the state $H|\psi\rangle$?