ASSIGNMENT 5 CMSC 858K (Fall 2015)

Due in class on Thursday, November 12.

1. *Spindles and pancakes.*

   (a) [2 points] Consider a map on density matrices that sends a state with Bloch vector (x, y, z) to one with Bloch vector $(0, 0, z)$. Show that this map is a quantum operation.

   (b) [3 points] Consider a map on density matrices that sends a state with Bloch vector (x, y, z) to one with Bloch vector $(x, y, 0)$. Is this map a quantum operation? Prove that your answer is correct.

2. *Effect of noise on state distinguishability.*

   Let $|\psi\rangle = |0\rangle$ and $|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$.

   (a) [2 points] Recall that the depolarizing channel with parameter $p \in [0, 1]$ is the quantum operation $\mathcal{D}_p(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$ on a qubit state ρ. Compute the action of the depolarizing channel on the states $|\psi\rangle$ and $|\phi\rangle$.

   (b) [4 points] Compute the trace distance between $\mathcal{D}_p(|\psi\rangle\langle\psi|)$ and $\mathcal{D}_p(|\phi\rangle\langle\phi|)$.

   (c) [1 point] Discuss how the depolarizing channel affects the distinguishability of quantum states.

3. *Properties of (relative) entropy.*

   (a) [3 points] The *relative entropy* (also known as the *Kullback-Leibler divergence*) of two probability distributions p and q is $D(p\|q) = \sum_i p_i \log(p_i/q_i)$. Prove that for any probability distributions p and q, we have $D(p\|q) \geq 0$.

   (b) [3 points] The relative entropy of two quantum states ρ and σ is $D(\rho\|\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma))$. Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ and $\sigma = \sum_i q_i|\phi_i\rangle\langle\phi_i|$ be spectral decompositions of ρ and σ, respectively. Show that

   $$D(\rho\|\sigma) = -S(\rho) - \sum_{i,j} p_i |\langle\psi_i|\phi_j\rangle|^2 \log q_j.$$

   (c) [3 points] Let $r_i = \sum_j |\langle\psi_i|\phi_j\rangle|^2 q_j$. Show that $D(\rho\|\sigma) \geq D(p\|r)$, and thereby conclude that $D(\rho\|\sigma) \geq 0$.

   (d) [2 points] Use nonnegativity of the relative entropy to show that if ρ is supported on a space of dimension d, then $S(\rho) \leq \log d$.

   (e) [3 points] Use nonnegativity of the relative entropy to prove the subadditivity of entropy, i.e., $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$. (Hint: consider the relative entropy of ρ_{AB} and $\rho_A \otimes \rho_B$.)

4. *Entanglement concentration.*

(a) [3 points] Suppose Alice has N qubits. For any $n \in \{0, 1, \dots, N\}$, define a projector

$$\Pi_n = \sum_{x:\, \mathrm{wt}(x)=n} |x\rangle\langle x|$$

onto the subspace of states with Hamming weight n, where the Hamming weight $\mathrm{wt}(x)$ is the number of 1s in the bit string x. Describe a quantum circuit that performs the projective measurement $\{\Pi_n\}$. (Note that Alice should not measure any refinement of this measurement, so superpositions of different x with the same Hamming weight remain coherent.)

(b) [3 points] Suppose Alice and Bob share N copies of the state $|\psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$, i.e., they have the state $|\psi\rangle^{\otimes N}$. Show that if Alice and Bob each perform the measurement from part (a), they will get the same outcome n, and show that the remaining state is equivalent, up to local unitary operations, to the maximally entangled state

$$\binom{N}{n}^{-1/2} \sum_{j=0}^{\binom{N}{n}-1} |j\rangle_A\,|j\rangle_B\,.$$

(c) [4 points] Let $\rho_A = \mathrm{Tr}_B\,|\psi\rangle\langle\psi|$. Show that for large N, the most likely value of n in the measurement from part (b) satisfies $\log\binom{N}{n} \approx NS(\rho_A)$. (In fact, for large N, the distribution is very highly peaked around this value of n. In other words, using only local operations and classical communication, Alice and Bob can convert N partially entangled states into about $NS(\rho_A)$ maximally entangled states with high probability.)