Practice Problems – Type Systems

Here is the simply typed lambda calculus, extended with integers, and its type system:

$$
\begin{array}{rcl}
e & ::= & v \mid x \mid e\ e \\
v & ::= & n \mid \lambda x{:}\,t.e \\
t & ::= & \mathit{int} \mid t \to t \\
A & ::= & \cdot \mid x : t, A
\end{array}
$$

$$
\textsc{Int}\ \frac{}{A \vdash n : \mathit{int}} \qquad
\textsc{Var}\ \frac{x \in \mathit{dom}(A)}{A \vdash x : A(x)} \qquad
\textsc{Lam}\ \frac{x{:}\,t, A \vdash e : t'}{A \vdash \lambda x{:}\,t.e : t \to t'} \qquad
\textsc{App}\ \frac{A \vdash e_1 : t \to t' \qquad A \vdash e_2 : t}{A \vdash e_1\ e_2 : t'}
$$

1. Draw derivations showing that the following typing judgments hold:

   (a) $\cdot \vdash 42 : \mathit{int}$

   (b) $y : \mathit{int} \vdash \lambda x{:}\,\mathit{int}.y : \mathit{int} \to \mathit{int}$

   (c) $\cdot \vdash \lambda x{:}\,\mathit{int}.\lambda y{:}\,\mathit{int}.x : \mathit{int} \to \mathit{int} \to \mathit{int}$

   (d) $+ : \mathit{int} \to \mathit{int} \to \mathit{int} \vdash (\lambda f{:}\,\mathit{int} \to \mathit{int}.f\ 42)\ (\lambda x{:}\,\mathit{int}. +\ x\ 3) : \mathit{int}$

   To save writing effort, you can write i instead of int.

2. Give a simply typed lambda calculus term that is type-incorrect, and yet it does not get stuck at run time. Your term must not be typable by the trivial operation of changing type annotations on parameters. For example, $(\lambda x{:}\,\mathit{int} \to \mathit{int}.x)\ 3$ is not a valid answer.

3. Finally, consider type inference for the simply-typed lambda calculus:

$$
\begin{array}{rcl}
e & ::= & v \mid x \mid e\ e \\
v & ::= & n \mid \lambda x.e \\
t & ::= & \alpha \mid \mathit{int} \mid t \to t \\
A & ::= & \cdot \mid x : t, A
\end{array}
$$

$$
\textsc{Int}\ \frac{}{A \vdash n : \mathit{int}} \qquad
\textsc{Var}\ \frac{x \in \mathit{dom}(A)}{A \vdash x : A(x)} \qquad
\textsc{Lam}\ \frac{x{:}\,\alpha, A \vdash e : t' \qquad \alpha \text{ fresh}}{A \vdash \lambda x.e : \alpha \to t'} \qquad
\textsc{App}\ \frac{\begin{array}{cc} A \vdash e_1 : t & A \vdash e_2 : t' \\ t = t' \to \beta & \beta \text{ fresh} \end{array}}{A \vdash e_1\ e_2 : \beta}
$$

   Draw derivations showing type inference applied to the following terms in the empty type environment; write down a solution to the associated constraints; and write down the fully resolved type of the term. We've done the first one as an example.

   (a) $(\lambda x.x)\ 42$

$$
\frac{\dfrac{x : \alpha \vdash x : \alpha}{\cdot \vdash \lambda x.x : \alpha \to \alpha} \qquad \cdot \vdash 42 : \mathit{int} \qquad (\alpha \to \alpha) = (\mathit{int} \to \beta)}{\cdot \vdash (\lambda x.x)\ 42 : \beta}
$$

   Solution: $\alpha = \beta = \mathit{int}$.

   Type: int.

   (b) $\lambda x.\lambda y.x$

   (c) $\lambda x.\lambda y.x\ y$

   (d) $(\lambda x.\lambda y.x)\ 3\ 42$