School Choice
and the Boston Mechanism

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School Choice Problem

College Admission Problem: Both schools and students are strategic agents (two-sided matching)

School Choice Problem: Only student welfare matters (one-sided matching)

- Schools do not choose priorities
- Similar to resource allocation

1974: In Boston, a court order forced racial integration of public schools via, leading to protests.

1981: Preempting a court order, Cambridge introduced the controlled choice program to...
- Increase diversity
- Treat students fairly
- Give families a choice

The Boston Mechanism was adopted by many US school districts.

1974: In Boston, a court order forced racial integration of public schools via, leading to protests.

1981: Preempting a court order, Cambridge introduced the controlled choice program to...
- Increase diversity
- Treat students fairly
- Give families a choice

The Boston Mechanism was adopted by many US school districts.

**Boston Mechanism:**

- **Assign as many students as possible to their first choice of school.**

**Round 1:**

- **choice:** For each school, assign students to seats based on priority.

  - :  
  - :  

**Round k:**

- **choice:** Consider each student’s kth choice:

  - For each school, assign students to seats based on priority.
What can we say about this mechanism?

- **Pareto Efficient**

- **Individually Rational** (if every student prefers being assigned to being unassigned)

- Strategy-proof?...

**Boston Mechanism:**

Assign as many students as possible to their first choice of school.

**Round 1:** For each school, assign students to seats based on priority.

Consider each student’s **first choice**: For each school, assign students to seats based on priority.

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</table>

**Round k:** For each school, assign students to seats based on priority. z
Some advice for strategic families:\n
“For a better chance of your ‘first choice’ school . . . consider choosing less popular schools.” (BPS School Guide)

“... find a school you like that is undersubscribed and put it as a top choice, OR, find a school that you like that is popular and put it as a first choice and find a school that is less popular for a ‘safe’ second choice.” (West Zone Parent Group)

Let’s consider the preference revelation game induced by the Boston Mechanism

Boston Mechanism:

Assign as many students as possible to their first choice of school.

Round 1: choice: For each school, assign students to seats based on priority.

Round k: For each school, assign students to seats based on priority.

Boston Game

**Students**

\[ I = \{i_1, i_2, \ldots, i_n\} \]

**Schools**

\[ S = \{s_1, s_2, \ldots, s_m\} \]

1: \{a > b > c\}

2: \{b > a > c\}

3: \{a > b > c\}

a: \{2, 1, 3\} | ( )

b: \{3, 1, 2\} | ( )

c: \{3, 2, 1\} | ( ) ( )
Boston Game

Students
$I = \{i_1, i_2, \ldots, i_n\}$

Schools
$S = \{s_1, s_2, \ldots, s_m\}$

Round 1:
- $1, 3$ propose to $a$ (only $1$ is accepted)
- $2$ proposes to $b$ (accepted)
Boston Game

Students

$I = \{i_1, i_2, \ldots, i_n\}$

Schools

$S = \{s_1, s_2, \ldots, s_m\}$

Round 1:
- 1,3 propose to a (only 1 is accepted)
- 2 proposes to b (accepted)

Round 2:
- 3 proposes to b (rejected)
Boston Game

Students
\( I = \{ i_1, i_2, \ldots, i_n \} \)

Schools
\( S = \{ s_1, s_2, \ldots, s_m \} \)

Round 1:
- 1,3 propose to a (only 1 is accepted)
- 2 proposes to b (accepted)

Round 2:
- 3 proposes to b (rejected)

Round 3:
- 3 proposes to c (accepted)

Is this a stable matching?
- Blocking pair: (3,b)
- 3 can improve assignment by playing \{ b > \ldots \}

Can we generalize?
The set of Nash equilibria of the Boston game is the set of stable matchings under the two-sided matching problem. (Ergin & Sonmez, 2006).

**Intuition:**
- If \((i, s_j)\) is a blocking pair, student \(i\) can guarantee a seat in school \(s_j\) by ranking it as her **top choice**.

Any stable matching \(\mu\) can be selected if all students request their assignment under this matching \(\mu(i)\) as their **top choice**.

What if some students don’t strategize?
What if some students don’t strategize?

Strategic Students

Sincere Students

$I_i^1$: Sincere students who rank school $i$ as first choice under and all strategic students

$I_i^2$: Sincere students who rank school $i$ as second choice

$\vdots$

$I_i^n$: Sincere students who rank school $i$ as $n^{th}$ choice
With sincere and strategic students, the Nash equilibria of the Boston game is the set of stable matchings under the augmented economy. (Pathak & Sonmez, 2008).

\[ \begin{align*}
1: \{a > b > c\} & \quad a: \{I_1^a, I_2^a, ..., I_3^a, 2, 1, 3\} \\
2: \{b > a > c\} & \quad b: \{I_1^b, I_2^b, ..., I_3^b, 3, 1, 2\} \\
& \quad \vdots \\
I_n: \{a > b > c\} & \quad s_m: \{I_1^m, I_2^m, ..., I_3^m, 3, 2, 1\}
\end{align*} \]

- \[ I_i^1: \text{Sincere students who rank school } i \text{ as first choice under and all strategic students} \]
- \[ I_i^2: \text{Sincere students who rank school } i \text{ as second choice} \]
- \[ I_i^n: \text{Sincere students who rank school } i \text{ as } n^{th} \text{ choice} \]
**Augmented Economy**

With sincere and strategic students, the Nash equilibria of the Boston game is the set of stable matchings under the augmented economy. (Pathak & Sonmez, 2008).

**Intuition:**
- For every school $s$, students who rank $s$ higher than other students effectively have higher priority at $s$.

(Def’n:) Strategic students select the Pareto-dominant Nash equilibrium

**Implications:**
- Sincere students’ assignments are the same under every Nash.
- If a sincere student $I$ becomes strategic:
  - Student $I$ weakly benefits
  - All other strategic students weakly suffer
Can we do better?

Yes! With the **student-optimal stable matching** (Gale & Shapley)

<table>
<thead>
<tr>
<th>Boston Game</th>
<th>Student-Optimal Mechanism</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leads to...</td>
<td>Leads to...</td>
</tr>
<tr>
<td>Pareto-dominant Nash equilibrium</td>
<td>Student-optimal stable matching</td>
</tr>
<tr>
<td>...which is preferred by</td>
<td>...which is preferred by</td>
</tr>
<tr>
<td>- All strategic</td>
<td>- All students</td>
</tr>
<tr>
<td>- Some sincere</td>
<td></td>
</tr>
</tbody>
</table>

**Deferred Acceptance Mechanism:**

Find the student-optimal stable matching

**Round 1:** Students propose their 1\textsuperscript{st} choice. Seats are *tentatively* accepted based on priority and capacity. Unassigned students are rejected.

: Unassigned students propose to their next choice. Full schools accept if proposers have higher priority than current students and reject otherwise.
Deferred Acceptance Mechanism:

Find the student-optimal stable matching

**Round 1:** Students propose their 1st choice. Seats are tentatively accepted based on priority and capacity. Unassigned students are rejected.

**Round k:** Unassigned students propose to their next choice. Full schools accept if proposers have higher priority than current students and reject otherwise.
Deferred Acceptance

Students

\[ I = \{i_1, i_2, \ldots, i_n\} \]

Schools

\[ S = \{s_1, s_2, \ldots, s_m\} \]

Round 1:
- 1 proposes to a (accepted)
- 2 proposes to b (accepted)
- 3 proposes to a (rejected)
Deferred Acceptance

Students
\[I = \{i_1, i_2, \ldots, i_n\}\]

Schools
\[S = \{s_1, s_2, \ldots, s_m\}\]

Round 1:
- 1 proposes to a (accepted)
- 2 proposes to b (accepted)
- 3 proposes to a (rejected)

Round 2:
- 3 proposes to b (accepted)
  - 2 is rejected

1: \(\{a > b > c\}\)
2: \(\{b > a > c\}\)
3: \(\{a > b > c\}\)
Deferred Acceptance

Students
\[ I = \{i_1, i_2, \ldots, i_n\} \]

Schools
\[ S = \{s_1, s_2, \ldots, s_m\} \]

Round 1:
- 1 proposes to a (accepted)
- 2 proposes to b (accepted)
- 3 proposes to a (rejected)

Round 2:
- 3 proposes to b (accepted)
  - 2 is rejected

Round 3:
- 2 proposes to a (accepted)
  - 1 is rejected
Deferred Acceptance

Students
\( I = \{ i_1, i_2, \ldots, i_n \} \)

Schools
\( S = \{ s_1, s_2, \ldots, s_m \} \)

Round 1:
- 1 proposes to a (accepted)
- 2 proposes to b (accepted)
- 3 proposes to a (rejected)

Round 2:
- 3 proposes to b (accepted)
  - 2 is rejected

Round 3:
- 2 proposes to a (accepted)
  - 1 is rejected

Round 3,4:
- 1 proposes to b (rejected)
- 1 proposes to c (accepted)
Deferred Acceptance

**Students**

$I = \{i_1, i_2, \ldots, i_n\}$

**Schools**

$S = \{s_1, s_2, \ldots, s_m\}$

**Deferred Acceptance:**
- Preferred by 2, 3

**Boston Mechanism:**
- Preferred by 1

Deferred acceptance is preferred by most non-strategic students...

But how much does this matter?
So what?

The core is small...

<table>
<thead>
<tr>
<th></th>
<th>20 percent</th>
<th>40 percent</th>
<th>60 percent</th>
<th>80 percent</th>
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<tbody>
<tr>
<td><strong>2005–2006</strong></td>
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<tr>
<td>Grade K2</td>
<td>0.14</td>
<td>0.08</td>
<td>0.04</td>
<td>0.01</td>
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<tr>
<td>Grade 6</td>
<td>0.38</td>
<td>0.20</td>
<td>0.07</td>
<td>0.01</td>
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<td><strong>2006–2007</strong></td>
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<tr>
<td>Grade K2</td>
<td>0.03</td>
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<tr>
<td>Grade 6</td>
<td>0.24</td>
<td>0.14</td>
<td>0.05</td>
<td>0.01</td>
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</table>

*Note:* This table is based on data provided by Boston Public Schools for Round 1 of their admissions process in 2005–2006 and 2006–2007.
Several reasons to use prefer a **strategy-proof mechanism**

- Strategies can be wrong (*inefficient*)
- Schools want *true* preference data
- Students waste time strategizing
- Boston Mechanism doesn’t guarantee a stable matching

But what if students *continue to strategize* under a strategy-proof mechanism?
So what?

The transition to strategy-proofness won’t rock the boat.

<table>
<thead>
<tr>
<th></th>
<th>Stated Choice</th>
<th>Boston Mechanism</th>
<th>Student-proposing</th>
<th>Top Trading Cycles Mechanism</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>number</td>
<td>percent</td>
<td>number</td>
<td>percent</td>
</tr>
<tr>
<td>Panel A: Elementary school applicants</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>2,590</td>
<td>77.9</td>
<td>2,451</td>
<td>73.7</td>
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<tr>
<td>2nd</td>
<td>399</td>
<td>9.3</td>
<td>419</td>
<td>12.6</td>
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<tr>
<td>3rd</td>
<td>103</td>
<td>3.1</td>
<td>173</td>
<td>5.2</td>
</tr>
<tr>
<td>4th</td>
<td>23</td>
<td>0.7</td>
<td>55</td>
<td>1.7</td>
</tr>
<tr>
<td>5th</td>
<td>12</td>
<td>0.4</td>
<td>23</td>
<td>0.7</td>
</tr>
<tr>
<td>Unassigned</td>
<td>289</td>
<td>8.7</td>
<td>205</td>
<td>6.2</td>
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<tr>
<td>Panel B: Middle school applicants</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>1st</td>
<td>4,197</td>
<td>77.3</td>
<td>3,922</td>
<td>72.2</td>
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<tr>
<td>2nd</td>
<td>417</td>
<td>7.7</td>
<td>701</td>
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<tr>
<td>3rd</td>
<td>269</td>
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<td>328</td>
<td>6.0</td>
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<td>4th</td>
<td>44</td>
<td>0.8</td>
<td>75</td>
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<td>5th</td>
<td>17</td>
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<td>Unassigned</td>
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<td>8.9</td>
<td>380</td>
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<tr>
<td>Panel C: High school applicants</td>
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<tr>
<td>1st</td>
<td>5,486</td>
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<td>5,261</td>
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<tr>
<td>2nd</td>
<td>407</td>
<td>6.4</td>
<td>624</td>
<td>9.8</td>
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<tr>
<td>3rd</td>
<td>158</td>
<td>2.5</td>
<td>236</td>
<td>3.7</td>
</tr>
<tr>
<td>4th</td>
<td>38</td>
<td>0.6</td>
<td>36</td>
<td>0.6</td>
</tr>
<tr>
<td>5th</td>
<td>6</td>
<td>0.1</td>
<td>7</td>
<td>0.1</td>
</tr>
<tr>
<td>Unassigned</td>
<td>285</td>
<td>4.5</td>
<td>216</td>
<td>3.4</td>
</tr>
</tbody>
</table>
Thank you!