LET’S TALK ABOUT PROJECTS
THIS CLASS: MATCHING & NOT THE NRMP
OVERVIEW OF THE NEXT 1.5 LECTURES

Stable marriage problem
- Bipartite, one vertex to one vertex

Stable roommates problem
- Not bipartite, one vertex to one vertex

Hospitals/Residents problem
- Bipartite, one vertex to many vertices
MATCHING WITHOUT INCENTIVES

Given a graph $G = (V, E)$, a matching is any set of pairwise non-adjacent edges

- No two edges share the same vertex
- Classical combinatorial optimization problem

Bipartite matching:

- Bipartite graph $G = (U, V, E)$
- Max cardinality/weight matching found easily – $O(VE)$ and better
  - E.g., through network flow, Hungarian algorithm, etc

Matching in general graphs:

- Also PTIME via Edmond’s algorithm – $O(V^2E)$ and better
STABLE MARRIAGE PROBLEM

Complete bipartite graph with equal sides:

- $n$ men and $n$ women (old school terminology 😞)

Each man has a strict, complete preference ordering over women, and vice versa

Want: a stable matching

Stable matching: No unmatched man and woman both prefer each other to their current spouses
## Example Preference Profiles

<table>
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<th>Albert</th>
<th>Diane</th>
<th>Emily</th>
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## EXAMPLE MATCHING #1

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Is this a stable matching?
Albert  |  Diane  |  Emily  |  Fergie
---      |  ---     |  ---     |  ---
Bradley  |  Emily   |  Diane   |  Fergie
Charles  |  Diane   |  Emily   |  Fergie

Diane  |  Bradley  |  Albert  |  Charles
---      |  ---     |  ---     |  ---
Emily  |  Albert  |  Bradley  |  Charles
Fergie  |  Albert  |  Bradley  |  Charles

No.
Albert and Emily form a blocking pair.
### Example Matching #2

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<tr>
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What about this matching?
### EXAMPLE MATCHING #2

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**Yes!**

(Fergie and Charles are unhappy, but helpless.)
SOME QUESTIONS

Does a stable solution to the marriage problem always exist?
Can we compute such a solution efficiently?
Can we compute the best stable solution efficiently?
GALE-SHAPLEY [1962]

1. Everyone is unmatched

2. While some man $m$ is unmatched:
   - $w := m$’s most-preferred woman to whom he has not proposed yet
   - If $w$ is also unmatched:
     - $w$ and $m$ are engaged
   - Else if $w$ prefers $m$ to her current match $m'$
     - $w$ and $m$ are engaged, $m'$ is unmatched
   - Else: $w$ rejects $m$

3. Return matched pairs
Claim
GS terminates in polynomial time (at most $n^2$ iterations of the outer loop)

Proof:
• Each iteration, one man proposes to someone to whom he has never proposed before
• $n$ men, $n$ women $\rightarrow n \times n$ possible events

(Can tighten a bit to $n(n - 1) + 1$ iterations.)
Claim
GS results in a perfect matching

Proof by contradiction:
• Suppose BWOC that $m$ is unmatched at termination
• $n$ men, $n$ women $\rightarrow w$ is unmatched, too
• Once a woman is matched, she is never unmatched; she only swaps partners. Thus, nobody proposed to $w$
• $m$ proposed to everyone (by def. of GS): $\Rightarrow$
Claim
GS results in a stable matching (i.e., there are no blocking pairs)

Proof by contradiction (1):
• Assume $m$ and $w$ form a blocking pair

Case #1: $m$ never proposed to $w$
• GS: men propose in order of preferences
• $m$ prefers current partner $w' > w$
• $\rightarrow m$ and $w$ are not blocking
Claim
GS results in a stable matching (i.e., there are no blocking pairs)

Proof by contradiction (2):
Case #2: $m$ proposed to $w$
- $w$ rejected $m$ at some point
- GS: women only reject for better partners
- $w$ prefers current partner $m' > m$
- $\rightarrow m$ and $w$ are not blocking

Case #1 and #2 exhaust space. $\Rightarrow$
RECAP: SOME QUESTIONS

Does a stable solution to the marriage problem always exist?  
We’ll look at a specific notion of “the best” – optimality with respect to one side of the market

Can we compute such a solution efficiently?  
We’ll look at a specific notion of “the best” – optimality with respect to one side of the market

Can we compute the best stable solution efficiently?
Let $S$ be the set of stable matchings

$m$ is a valid partner of $w$ if there exists some stable matching $S$ in $S$ where they are paired

A matching is man optimal (resp. woman optimal) if each man (resp. woman) receives their best valid partner

• Is this a perfect matching? Stable?

A matching is man pessimal (resp. woman pessimal) if each man (resp. woman) receives their worst valid partner
Claim
GS – with the man proposing – results in a man-optimal matching

Proof by contradiction (1):
• Men propose in order \( \rightarrow \) at least one man was rejected by a valid partner
• Let \( m \) and \( w \) be the first such reject in \( S \)
• This happens because \( w \) chose some \( m' > m \)
• Let \( S' \) be a stable matching with \( m, w \) paired
  \( (S' \) exists by def. of valid)
Claim
GS – with the man proposing – results in a man-optimal matching

Proof by contradiction (2):
• Let $w'$ be partner of $m'$ in $S'$
• $m'$ was not rejected by valid woman in $S$ before $m$ was rejected by $w$ (by assump.)
  $\rightarrow m'$ prefers $w$ to $w'$
• Know $w$ prefers $m'$ over $m$, her partner in $S'$
  $\rightarrow m'$ and $w$ form a blocking pair in $S'$
Does a stable solution to the marriage problem always exist? Yes

Can we compute such a solution efficiently? Yes

Can we compute the best stable solution efficiently? Yes

For one side of the market. What about the other side?
Claim
GS – with the man proposing – results in a woman-pessimal matching

Proof by contradiction:
• $m$ and $w$ matched in $S$, $m$ is not worst valid
• $\rightarrow$ exists stable $S'$ with $w$ paired to $m' < m$
• Let $w'$ be partner of $m$ in $S'$
• $m$ prefers to $w$ to $w'$ (by man-optimality)
• $\rightarrow m$ and $w$ form blocking pair in $S'$ $>>$
INCENTIVE ISSUES

Can either side benefit by misreporting?

• (Slight extension for rest of talk: participants can mark possible matches as unacceptable – a form of preference list truncation)

Any algorithm that yields woman-(man-)optimal matching

→

truthful revelation by women (men) is dominant strategy [Roth 1982]
In GS with men proposing, women can benefit by misreporting preferences.

### Truthful reporting

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### Strategic reporting

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Claim

There is no matching mechanism that:
1. is strategy proof (for both sides); and
2. always results in a stable outcome (given revealed preferences)
EXTENSIONS TO STABLE MARRIAGE
What if we have $n$ men and $n' \neq n$ women?

How does this affect participants? Core size?

- Being on short side of market: good!
- W.h.p., short side get rank $\sim \log(n)$
- … long side gets rank $\sim$random

# women held constant at $n' = 40$
Not many stable matchings with even small imbalances in the market
“Rural hospital theorem” [Roth 1986]:

- The set of residents and hospitals that are unmatched is the same for all stable matchings

Assume $n$ men, $n+1$ women

- One woman $w$ unmatched in all stable matchings
- $\rightarrow$ Drop $w$, same stable matchings

Take stable matchings with $n$ women

- Stay stable if we add in $w$ if no men prefer $w$ to their current match
- $\rightarrow$ average rank of men’s matches is low
ONLINE ARRIVAL [KHUDLER ET AL. 1993]

Random preferences, men arrive over time, once matched nobody can switch

Algorithm: match \( m \) to highest-ranked free \( w \)

- On average, \( O(n \log(n)) \) unstable pairs

No deterministic or randomized algorithm can do better than \( \Omega(n^2) \) unstable pairs!

- Not better with randomization 😞
INCOMPLETE PREFS
[MANLOVE ET AL. 2002]

Before: complete + strict preferences
- Easy to compute, lots of nice properties

Incomplete preferences
- May exist: stable matchings of different sizes

Everything becomes hard!
- Finding max or min cardinality stable matching
- Determining if $<m,w>$ are stable
- Finding/approx. finding “egalitarian” matching
NON-BIPARTITE GRAPH ...?

Matching is defined on general graphs:

• “Set of edges, each vertex included at most once”
• (Finally, no more “men” or “women” …)

The stable roommates problem is stable marriage generalized to any graph

Each vertex ranks all n-1 other vertices

• (Variations with/without truncation)

Same notion of stability
IS THIS DIFFERENT THAN STABLE MARRIAGE?

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<tr>
<th></th>
<th>Alana</th>
<th>Brian</th>
<th>Cynthia</th>
<th>Dracula</th>
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<td>Alana</td>
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<td>Dracula</td>
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No stable matching exists!

Anyone paired with Dracula (i) prefers some other $v$ and (ii) is preferred by that $v$. 
HOPELESS?

Can we build an algorithm that:

- Finds a stable matching; or
- Reports nonexistence

... In polynomial time?

Yes! [Irving 1985]

- Builds on Gale-Shapley ideas and work by McVitie and Wilson [1971]
IRVING’S ALGORITHM: PHASE 1

Run a deferred acceptance-type algorithm

If at least one person is unmatched: nonexistence

Else: create a reduced set of preferences

• $a$ holds proposal from $b$ \rightarrow $a$ truncates all $x$ after $b$
• Remove $a$ from $x$’s preferences
• Note: $a$ is at the top of $b$’s list

If any truncated list is empty: nonexistence

Else: this is a “stable table” – continue to Phase 2
STABLE TABLES

1. $a$ is first on $b$’s list iff $b$ is last on $a$’s

2. $a$ is not on $b$’s list iff
   - $b$ is not on $a$’s list
   - $a$ prefers last element on list to $b$

3. No reduced list is empty

Note 1: stable table with all lists length 1 is a stable matching

Note 2: any stable subtable of a stable table can be obtained via rotation eliminations
IRVING’S ALGORITHM: PHASE 2

Stable table has length 1 lists: return matching

Identify a rotation:

\((a_0, b_0), (a_1, b_1), \ldots, (a_{k-1}, b_{k-1})\) such that:
- \(b_i\) is first on \(a_i\)'s reduced list
- \(b_{i+1}\) is second on \(a_i\)'s reduced list (\(i+1\) is mod \(k\))

Eliminate it:
- \(a_0\) rejects \(b_0\), proposes to \(b_1\) (who accepts), etc.

If any list becomes empty: nonexistence
If the subtable hits length 1 lists: return matching
Claim
Irving’s algorithm for the stable roommates problem terminates in polynomial time – specifically $O(n^2)$.

This requires some data structure considerations

- Naïve implementation of rotations is $\sim O(n^3)$
ONE-TO-MANY MATCHING

The hospitals/residents problem (aka college/students problem aka admissions problem):
• Strict preference rankings from each side
• One side (hospitals) can accept $q > 1$ residents

Also introduced in [Gale and Shapley 1962]

Has seen lots of traction in the real world
• Candice will tell you about some of this next lecture!
• 11/15 and 11/17 – will talk about school choice
NEXT CLASS:
CANDICE SCHUMANN

LIDA APERGI
BERTSIMAS, FARIAS, AND TRICHAKIS. FAIRNESS, EFFICIENCY, AND FLEXIBILITY IN ORGAN ALLOCATION FOR KIDNEY TRANSPLANTATION. OPERATIONS RESEARCH, 2013.