Axioms of Quantum

- States
- Operations
- Measurements
- (composite systems)

States

- (in randomized computation, described system using a vector, similar)

A q-state is a unit vector in a Hilbert Space. (Almost always \( \mathbb{C}^N \), will use \( \mathbb{C} \) if \( N \) not specified)

- N-dim state

\[
\left( \begin{array}{c}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{N-1} \\
\end{array} \right)
\]

\( a_i \in \mathbb{C} \) - "amplitudes"

Unit Normalization:

\[
\sum_{i=0}^{N-1} |a_i|^2 = 1
\]

- Similar to probability vectors except instead of adding to 1, absolute values squared add to 1

\[
\left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{i}{\sqrt{2}} \right|^2 = \frac{1}{2} + \frac{1}{2} = 1
\]

- Say state is "normalized"

Notation (bra, ket)

- \( | \psi \rangle \) = "ket" = q-state vector

- \( \langle \psi | \) "state \( \psi \), "\( \psi \)" ket \( \psi \)" = \( \bar{\psi} \)

Computational / Standard basis states:

\( |0\rangle, |1\rangle, |2\rangle \ldots \)

\[
|0\rangle = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ \vdots \end{array} \right), \quad
|1\rangle = \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ \vdots \end{array} \right) \ldots
\]

- Not in superposition

\[
|\psi\rangle = \sum_{i=0}^{N-1} a_i |i\rangle = \left( \begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{array} \right)
\]

- Three ways to represent the same q-state

\[
\langle \psi | \psi \rangle = 1
\]

Annoying to write out whole vector, so use ket to symbolize.
\[ \langle \psi | = \text{"bra"} \]
\[ \langle \psi | \phi \rangle \rightarrow \langle \psi | \phi \rangle = (a_0^* \ldots a_{N-1}^*) \begin{pmatrix} b_0 \\ \vdots \\ b_{N-1} \end{pmatrix} = \sum_{i=0}^{N-1} a_i^* b_i \in \mathbb{C} \]

\[ \langle \psi | \psi \rangle = \sum_{i=0}^{N-1} a_i^* a_i = 1 \]
\[ \langle \psi | \phi \rangle = 0 \iff \text{Orthogonal} \]

\[ \text{\"ket-bra\"} \]
\[ \langle \psi | \phi \rangle \rightarrow | \psi \rangle \langle \psi | \phi \rangle \]

\[ \text{can be different dimensions} \]

**Example:**
\[ C^2 : |0\rangle = (1) \quad |1\rangle = (0) \]
\[ 10 \times 10 = |10\rangle = (1) \langle 0 | = (0) \]
\[ 10 \times |1\rangle = (1) \langle 0 | = (0) \langle 1 | = (0, 1) \]

**Active:**
Find normalized states
\[ |\psi\rangle = \left(\frac{1}{\sqrt{2}}\right) \quad |\phi\rangle = \left(\frac{1}{\sqrt{2}}\right) \]
\[ |\psi\rangle, |\phi\rangle \quad \text{s.t.} \quad \langle \phi | \phi \rangle = \langle \psi | \psi \rangle = 0 \]

**Labels:** (special states in \( C^2 \))
\[ |0\rangle = (1) \quad |1\rangle = (0) \quad |+\rangle = \frac{1}{\sqrt{2}}(1) \quad |-\rangle = \frac{1}{\sqrt{2}}(-1) \quad |\uparrow\rangle = \frac{1}{\sqrt{2}}(1) \quad \downarrow = \frac{1}{\sqrt{2}}(-1) \]
Operations/Gates

In randomized computing, we saw state of the system is a vector that is transformed by a left stochastic matrix. Here, too, state is transformed by a matrix...

Time evolution of an isolated q. state in $\mathcal{H}$

$$|\psi\rangle \in \mathbb{C}^n \quad U \in \mathbb{C}^{n \times n}$$

is given by a unitary operator acting on $\mathcal{H}$

$$U \text{ is unitary } \iff U^\dagger = U^{-1} \iff uu^\dagger = uu^\dagger = 1$$

Want: If $|\psi\rangle$ is valid q. state, $|\psi\rangle = U|\psi\rangle$

Need: $\langle\psi'|\psi\rangle = 1$

$$\langle\psi'| = (U|\psi\rangle)^\dagger = (1|\psi\rangle)^+ U^\dagger = \langle\psi|U^\dagger$$

$$\langle\psi'|\psi\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle = 1$$

---

ex: Reversible gates from Randomized Computing

Matrix Rep

**NOT**

$$\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}$$

S.B.

Transformation

$|0\rangle \rightarrow |1\rangle$

$|1\rangle \rightarrow |0\rangle$

Ket-bra

$$\sum_{i,j} u_{ij}|i\rangle|j\rangle$$

$$U = \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)$$

CNOT:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$|0\rangle \rightarrow |+\rangle$

$|1\rangle \rightarrow |1\rangle$

$|\psi\rangle = |0\rangle |x\rangle$ (Phase)

Ket-bra:

$$\begin{array}{l}
|0\rangle|1\rangle + |1\rangle|0\rangle \\
|0\rangle|1\rangle + |1\rangle|0\rangle
\end{array}$$

More Exotic

$|0\rangle|1\rangle + |1\rangle|0\rangle$ (Phase)

$|0\rangle|1\rangle + |1\rangle|0\rangle$ (Phase)

Matrix Rep

$U (\sum_{i=0}^{N-1} a_i|i\rangle) = \sum_{i=0}^{N-1} a_i U|i\rangle$

S.B. affect:

$$U (\sum_{i=0}^{N-1} a_i|i\rangle) = \sum_{i=0}^{N-1} a_i (U|i\rangle)$$

Ket-bra:

$$U (\sum_{i=0}^{N-1} a_i|k\rangle) = \sum_{i=0}^{N-1} a_i (U|k\rangle)$$

Who knows K.B.?
Remarks

- Unitaries are reversible! \( U^{-1} = U^* \) but \( U^* \) is a unitary (b/c \( U^*U = \mathbb{I} \)), so can apply \( U^* \) to reverse \( U \).

- Unitaries transform orthonormal bases

  Orthonormal basis \( B = \{ |\psi_1\rangle, \ldots, |\psi_n\rangle \} \) s.t. \( |\psi_i\rangle \in \mathbb{C}^n \), \( \langle \psi_i | \psi_j \rangle = \delta_{ij} \).

  If \( B \) is orthonormal basis then \( UB = \{ U|\psi_1\rangle, \ldots, U|\psi_n\rangle \} \) is orthonormal.

- Multiple unitaries in a row: \( U_k U_{k-1} \cdots U_2 U_1 = U \)

  \( \text{normal matrix multiplication} \) \( \uparrow \) order goes right to left in order of application

Measurements (von Neumann)

Let \( M = \{ |\phi\rangle \} \) be an O.B. for \( \mathcal{H} \). Then if measure \( |\psi\rangle \in \mathcal{H} \) with \( M \), with probability \( |\langle \phi | \psi \rangle|^2 \):

- get outcome "i" and,
- state becomes \( |\phi\rangle \) "projects" "collapses".

No way to get information out of state except by measuring (can't access amplitudes) but measuring destroys state

\[
|\psi\rangle = \sum_{i=0}^{N-1} a_i |i\rangle, \quad M = \{ |j\rangle \}_{i=0}^{N-1}
\]

\[
|\langle j | \psi \rangle|^2 = \sum_{i=0}^{N-1} a_i^* \langle j | i \rangle = a_j \rightarrow \text{w/p } |a_j|^2 \text{ get } "j", \text{ project to } |j\rangle
\]