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Axioms of Quantum

- States
- operations
- measurements
- (· composite systems)

- Axioms proscribe how to mathematically describe aspects of q. mechanics
- Mathematical description aligns perfectly with our experimental understanding (up to GR)

States

· (in Randomized computation, described system using a vector, similar).

A q. state is a unit vector in a Hilbert Space. (Almost always $\mathbb{C}^N$, will use $\mathcal{H}$ if N not specified)

- N-dim state $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$

$a_i \in \mathbb{C}$ ← "amplitudes"

Unit Normalization: $\sum_{i=0}^{N-1} |a_i|^2 = 1$

- Similar to probability vectors except instead of adding to 1, absolute values squared add to 1

- ex: $N=2$ $\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$

$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1$

- Say state is "normalized"

Notation (bra, ket)

annoying to write out whole vector, so use ket to symbolize

- $|\cdot\rangle$ = "ket" = q. state vector

$|\psi\rangle$ "state psi", "psi", "ket psi" $\approx \vec{\psi}$

Computational / Standard basis states:

$|0\rangle, |1\rangle, |2\rangle \ldots$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ... ← Not in superposition

← In Superposition

$|\psi\rangle = \sum_{i=0}^{N-1} a_i |i\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$ ⇐ Three ways to represent the same q. state

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- $\langle\cdot| =$ "<u>bra</u>"

$\langle\psi| = (|\psi\rangle)^{\dagger}$ ← conjugate transpose

so if $|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$ then $\langle\psi| = (a_0^*, a_1^*, a_2^*, \ldots)$

- "<u>bra-ket</u>"

$|\psi\rangle, |\phi\rangle \longrightarrow \langle\psi|\phi\rangle = (a_0^* \cdots a_{N-1}^*)\begin{pmatrix} b_0 \\ \vdots \\ b_{N-1} \end{pmatrix} = \sum_{i=0}^{N-1} a_i^* b_i \in \mathbb{C}$

↑ ↗ same dimension

$\langle\psi|\psi\rangle = \sum_{i=0}^{N-1} a_i^* a_i = 1$

$\boxed{ex}$: $\langle i|j\rangle = \delta_{ij}$

$\langle\psi|\phi\rangle = 0$ ⇐ Orthogonal

- "<u>ket-bra</u>"

$|\psi\rangle, |\phi\rangle \longrightarrow |\psi\rangle\langle\phi|$

$|\psi\rangle\langle\phi|$

↑ ↗ can be different dimensions

<u>ex</u>: $\mathbb{C}^2$: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|0\rangle\langle 1| = |0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0\ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Active

$|\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $|\phi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ Find normalized states $|\psi^{\perp}\rangle, |\phi^{\perp}\rangle$ s.t. $\langle\phi|\phi^{\perp}\rangle = \langle\psi|\psi^{\perp}\rangle = 0$

Labels: (special states in $\mathbb{C}^2$)

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $|\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ $|\leftarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

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Operations/Gates

In randomized computing, we saw state of the system is a vector that is transformed by a left stochastic matrix. Here, too, state is transformed by a matrix...

Time evolution of an isolated q. state in $\mathcal{H}$ is given by a unitary operator acting on $\mathcal{H}$ $\left(|\psi\rangle \in \mathbb{C}^N \quad U \in \mathbb{C}^N \times \mathbb{C}^N \right)$

$$U \text{ is unitary} \iff U^\dagger = U^{-1} \iff UU^\dagger = U^\dagger U = \mathbb{I}$$

Want: If $|\psi\rangle$ is valid q. state, $|\psi'\rangle = U|\psi\rangle$ " "

← Need $\langle\psi'|\psi'\rangle = 1$

$$\langle\psi'| = \left(U|\psi\rangle\right)^\dagger = \left(|\psi\rangle\right)^\dagger U^\dagger = \langle\psi|U^\dagger$$

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle = 1$$

ex:

	• Reversible gates from Randomized Computing		• More Exotic
	"NOT"	CNOT	
Matrix Rep	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
S.B. Transformation	$\|0\rangle \to \|1\rangle$ $\|1\rangle \to \|0\rangle$	$\|0\rangle \to \|0\rangle$ $\|1\rangle \to \|1\rangle$ $\|2\rangle \to \|3\rangle$ $\|3\rangle \to \|2\rangle$	*Phases are important * $\|0\rangle \to \|+\rangle$, $\|1\rangle \to \|-\rangle$ $\neq$ $\|0\rangle \to \|+\rangle$, $\|1\rangle \to -\|-\rangle$
Ket-bra $\sum_{i,j=1}^{N-1} u_{ij}\|i\rangle\langle j\|$ $U = \left(\longrightarrow \overset{\downarrow}{u_{ij}}\right)$	$\|0\rangle\langle 1\| + \|1\rangle\langle 0\|$	* $\|0\rangle\langle 0\| + \|1\rangle\langle 1\| + \|3\rangle\langle 2\| + \|2\rangle\langle 3\|$	$\frac{1}{\sqrt{2}}\left(\|0\rangle\langle 0\| + \|0\rangle\langle 1\| + \|1\rangle\langle 0\| - \|1\rangle\langle 1\|\right)$

Matrix Rep: $\left(U \right)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}$

S.B. affect: $U\left(\sum_{i=0}^{N-1} a_i|i\rangle\right) = \sum_{i=0}^{N-1} a_i\left(U|i\rangle\right)$

Ket-bra: $U\left(\sum_{i=0}^{N-1} a_i|i\rangle\right) = \sum_{i=0}^{N-1} a_i\left(\sum_{kj} u_{kj}|k\rangle\underbrace{\langle j||i\rangle}_{\delta_{ij}\ \to j=i}\right) = \sum_{i=0,\,k=0}^{N-1} a_i u_{ki}|k\rangle$

who knows K.D.?

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Remarks

- Unitaries are reversible! $U^{-1} = U^{\dagger}$ but $U^{\dagger}$ is a unitary.
  (b/c $U^{\dagger}U = \mathbb{I}$), so can apply $U^{\dagger}$ to reverse U

- Unitaries transform orthonormal bases
  Orthonormal basis $B = \{|\psi_1\rangle, \cdots, |\psi_N\rangle\}$ s.t. $|\psi_i\rangle \in \mathbb{C}^N$, $\langle\psi_i|\psi_j\rangle = \delta_{ij}$
  If B is orthonormal basis then $U \cdot B = \{U|\psi_1\rangle, \ldots U|\psi_N\rangle\}$ is orthonormal

- Multiple unitaries in a row: $U_k U_{k-1} \cdots U_2 U_1 = U$
  normal matrix multiplication
  order goes right to left in order of application

Measurements (von Neumann)

Let $M = \{|\phi_i\rangle\}$ be an O.B. for $\mathcal{H}$. Then if measure $|\psi\rangle \in \mathcal{H}$ with M,
with probability $|\langle\phi_i|\psi\rangle|^2$:
- get outcome "i" and,
- state becomes $|\phi_i\rangle$ "projects" "collapses"

No way to get information out of state except by measuring, (can't access amplitudes) but measuring destroys state

ex: $|\psi\rangle = \sum_{i=0}^{N-1} a_i |i\rangle$, $M = \{|j\rangle\}_{i=0}^{N-1}$

similar to probability vector!

$\langle j|\psi\rangle = \sum_{i=0}^{N-1} a_i \langle j|i\rangle = a_j \longrightarrow$ w/p $|a_j|^2$ get "j", project to $|j\rangle$