CMSC 451:Fall 2017 Dave Mount

Homework 5: NP-Completeness

Handed out Thu, Nov 30. **Due:** Monday, Dec 11, 11:59pm (electronic submission through ELMS.)

Problem 1: Highly intelligent aliens land on Earth and tell us the following two things and then leave immediately.

(a) The 3-Coloring problem (which is NP-complete) is solvable in worst-case $O(n^9)$ time, where n denotes the number of vertices in the graph.

(b) There is no algorithm for 3-Coloring that runs faster than $\Omega(n^7)$ time in the worst case.

Assuming these two facts, for each of the following assertions, indicate whether it can be inferred from the information the aliens have given us. (In all cases, time complexities are understood to be *worst-case* running time.) Provide a short justification in each case.

(i) All *NP-complete* problems are solvable in polynomial time.

(ii) All problems in NP, even those that are *not* NP-complete, are solvable in polynomial time.

(iii) All *NP-hard* problems are solvable in polynomial time.

(iv) All NP-complete problems are solvable in $O(n^9)$ time.

(v) No NP-complete problem can be solved faster than $\Omega(n^7)$ time.

Problem 2. Recall that a boolean formula is said to be in *conjunctive normal form* (CNF), if it is the logical-and of a set of clauses, where each clause is the logical-or of a set of literals, and each literal is a variable or its negation. Suppose that you are given a boolean formula F in CNF (where each clause may one, two, three, or more literals). An example of such a formula is shown below.

$$F = (a \vee b \vee \overline{c} \vee \overline{d}) \wedge (e \vee \overline{b}) \wedge (d) \wedge (\overline{a} \vee e \vee f \vee \overline{g} \vee \overline{c}).$$

(a) Explain how to convert any such formula F into an *equivalent* 3-CNF formula F'. This means that F' has *exactly* three literals per clause, and F' is satisfiable if and only if F is satisfiable. Your conversion should run in polynomial time. Briefly justify the correctness of your construction.

Hint: A 3-CNF clause is allowed to contain multiple copies of the same literal. Your conversion process is allowed to create new variables.

(b) Show the result of your transformation on the specific formula F above.

(c) Can you further enhance your transformation to convert any CNF formula in polynomial time to an equivalent one in 2-CNF (exactly two literals per clause)? If possible, explain how to do this. If not, explain what goes wrong when you try to adapt your solution to part (a).

Problem 3. Given a directed graph $G = (V, E)$, a *vertex cycle cover* is a subset of vertices such that every simple cycle in G passes through at least one of these vertices. (For example, the graph shown in Fig 1 has a vertex cycle cover of size 2 (shaded).)

Vertex Cycle Cover (VCC): Given a digraph G and an integer k, does G contain a vertex cycle cover of size at most k?

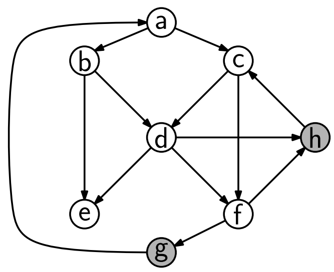


Figure 1: Problem 3: A digraph and a vertex cycle cover consisting of $\{g, h\}$.

Show that VCC is in NP. (You do *not* need to show that it is NP-hard, but see the challenge problem.)

Hint: Be careful. A graph can have exponentially many simple cycles, so a naive implementation will take exponential time.

Problem 4. When finding cliques, it is natural to look for vertices of high degree. Suppose, however, that you want to find cliques consisting of vertices of relatively low degree. We will show that even this problem is NP-complete.

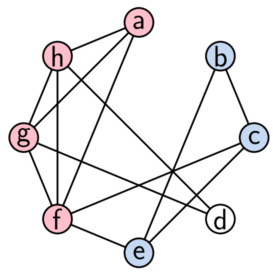
Low-Degree Clique (LDC): Given a graph $G = (V, E)$ and an integer k, does G have a clique of size at least k consisting entirely of vertices who degree is not greater than the median vertex degree of the entire graph?

By the *median vertex degree*, we mean the median value of the degrees of all n vertices of the graph. (For our purposes, we define the *median* of a set of n numbers to be the $\lceil n/2 \rceil$-smallest value of the set.)

For example, the graph shown in Fig. 2 has median vertex degree of 3. There exists an LDC of size 3 (vertices $\{b, c, e\}$) since all these vertices have degree at most 3. Even though there is a clique of size 4 (vertices $\{a, f, g, h\}$) it is not an LDC since it contains (at least one) vertex of degree higher than 3.

Show that the LDC problem is NP-Complete. Remember that this involves two things. Showing that LDC is in NP, and that some known NP-complete problem is polynomial time reducible to LDC.

Hint: Reduction from the standard Clique problem.



vertex:	a	b	c	d	e	f	g	h
degree:	3	2	3	2	3	5	4	4
median degree:	3							

Figure 2: Problem 4: Low-Degree Clique.

Challenge Problem 1.

(Remember that challenge problems count for extra credit points. These additional points are factored in only after the final cutoffs have been set, and can only increase your final grade.)

In reference to problem (3), show that the VCC problem is NP-complete, by showing that some known NP-complete problem can be reduced to it. (**Hint:** Reduction from either Vertex Cover or Independent Set.)

Challenge Problem 2. The requirements for the reduction given in Problem 4 do not forbid the existence of cliques (even very large ones) whose vertices have degree *larger* than the median degree. Show that there exists a reduction to the LDC problem, where the largest clique in the output of the transformation does not exceed the size of the largest clique in the original graph.