CMSC 451: Lecture 21
NP-Completeness: Clique, Vertex Cover, and Dominating Set
Thursday, Nov 30, 2017

Reading: DPV Section 8.3, KT Section 8.4. Dominating set is not given in either text.

Recap: Last time we gave a reduction from 3SAT (satisfiability of boolean formulas in 3-CNF form) to IS (independent set in graphs). Today we give a few more examples of reductions. Recall that to show that a decision problem (language) L is NP-complete we need to show:

(i) $L \in$ NP. (That is, given an input and an appropriate certificate, we can guess the solution and verify whether the input is in the language), and

(ii) L is NP-hard, which we can show by giving a reduction from some known NP-complete problem L' to L, that is, $L' \leq_P L$. (That is, there is a polynomial time function that transforms an instance L' into an equivalent instance of L for the other problem).

Some Easy Reductions: Next, let us consider some closely related NP-complete problems:

Clique (CLIQUE): The *clique problem* is: given an undirected graph $G = (V, E)$ and an integer k, does G have a subset V' of k vertices such that for each distinct $u, v \in V'$, $\{u, v\} \in E$. In other words, does G have a k vertex subset whose induced subgraph is complete.

Vertex Cover (VC): A *vertex cover* in an undirected graph $G = (V, E)$ is a subset of vertices $V' \subseteq V$ such that every edge in G has at least one endpoint in V'. The *vertex cover problem* (VC) is: given an undirected graph G and an integer k, does G have a vertex cover of size k?

Dominating Set (DS): A *dominating set* in a graph $G = (V, E)$ is a subset of vertices V' such that every vertex in the graph is either in V' or is adjacent to some vertex in V'. The *dominating set problem* (DS) is: given a graph $G = (V, E)$ and an integer k, does G have a dominating set of size k?

Don't confuse the clique (CLIQUE) problem with the clique-cover (CC) problem that we discussed in an earlier lecture. The clique problem seeks to find a single clique of size k, and the clique-cover problem seeks to partition the vertices into k groups, each of which is a clique.

We have discussed the facts that cliques are of interest in applications dealing with clustering. The vertex cover problem arises in various servicing applications. For example, you have a compute network and a program that checks the integrity of the communication links. To save the space of installing the program on every computer in the network, it suffices to install it on all the computers forming a vertex cover. From these nodes all the links can be tested. Dominating set is useful in facility location problems. For example, suppose we want to select where to place a set of fire stations such that every house in the city is within two minutes of the nearest fire station. We create a graph in which two locations are adjacent if they are within two minutes of each other. A minimum sized dominating set will be a minimum set of locations such that every other location is reachable within two minutes from one of these sites.

The CLIQUE problem is obviously closely related to the independent set problem (IS): Given a graph G does it have a k vertex subset that is completely *disconnected*. It is not quite as clear that the vertex cover problem is related. However, the following lemma makes this connection clear as well (see Fig. 1). Given a graph G, recall that $\overline{G}$ is the *complement graph* where edges and non-edges are reverse. Also, recall that $A \setminus B$ denotes set resulting by removing the elements of B from A.

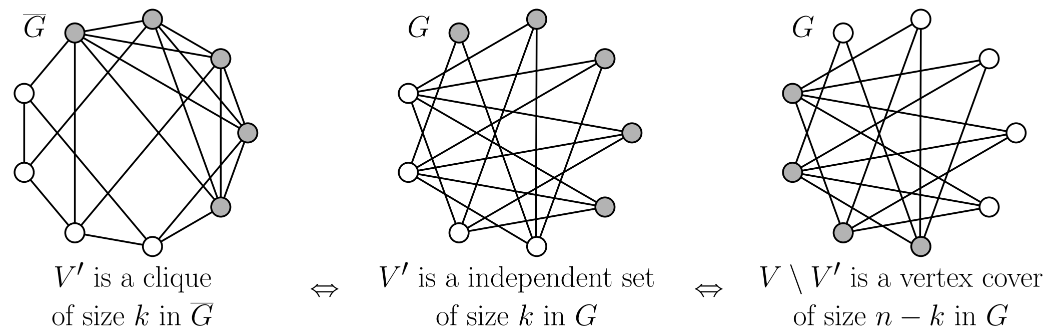

Fig. 1: Clique, Independent set, and Vertex Cover.

Lemma: Given an undirected graph $G = (V, E)$ with n vertices and a subset $V' \subseteq V$ of size k. The following are equivalent:

(i) V' is a clique of size k for the complement, $\overline{G}$

(ii) V' is an independent set of size k for G

(iii) $V \setminus V'$ is a vertex cover of size $n - k$ for G, (where $n = |V|$)

Proof:

(i) ⇒ (ii): If V' is a clique for $\overline{G}$, then for each $u, v \in V'$, $\{u, v\}$ is an edge of $\overline{G}$ implying that $\{u, v\}$ is not an edge of G, implying that V' is an independent set for G.

(ii) ⇒ (iii): If V' is an independent set for G, then for each $u, v \in V'$, $\{u, v\}$ is not an edge of G, implying that every edge in G is incident to a vertex in $V \setminus V'$, implying that $V \setminus V'$ is a vertex cover for G.

(iii) ⇒ (i): If $V \setminus V'$ is a vertex cover for G, then for any $u, v \in V'$ there is no edge $\{u, v\}$ in G, implying that there is an edge $\{u, v\}$ in $\overline{G}$, implying that V' is a clique in $\overline{G}$.

Thus, if we had an algorithm for solving any one of these problems, we could easily translate it into an algorithm for the others. In particular, we have the following.

Theorem: CLIQUE is NP-complete.

CLIQUE ∈ NP: Given an instance (G, k) for CLIQUE, we guess the k vertices that will form the clique. (These vertices form the *certificate*.) We can easily verify in polynomial time that all pairs of vertices in the set are adjacent (e.g., by inspection of $O(k^2)$ entries of the adjacency matrix). If so, we output "yes" and otherwise "no". (If the graph has a CLIQUE of size k, one of these guesses will work, and so we will correctly classify G as having a clique of size k. If not, all guesses will fail, and we will correctly classify G as not having such a clique.)

IS $\leq_P$ CLIQUE: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the CLIQUE problem in polynomial time. The reduction function f inputs G and k, and outputs the pair $(\overline{G}, k)$. Clearly this can be done in polynomial time. By the above lemma, this instance is equivalent.

Theorem: VC is NP-complete.

VC $\in$ NP: Given an instance (G, k) for VC, we guess the k vertices that will form the vertex cover. (Again, these vertices form the *certificate*.) We then verify that these vertices form a vertex cover, by checking that every edge of G is incident to one of these vertices. If so, we output "yes" and otherwise "no". (Again, if G has a vertex cover of size k, one of these guesses will work, and we correctly classify G as having a vertex cover of size k. Otherwise, all fail and we classify G as not having such a vertex cover.)

IS $\leq_P$ VC: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the VC problem in polynomial time. The reduction function f inputs G and k, computes the number of vertices, n, and then outputs $(G, n - k)$. Clearly this can be done in polynomial time. By the lemma above, these instances are equivalent.

Note: Note that in each of the above reductions, the reduction function did not know whether G has an independent set or not. It must run in polynomial time, and IS is an NP-complete problem. So it does not have time to determine whether G has an independent set or which vertices are in the set.

Dominating Set: As with vertex cover, dominating set is an example of a graph covering problem. Here the condition is a little different, each *vertex* is *adjacent* to at least one member of the dominating set, as opposed to each *edge* being *incident* to at least one member of the vertex cover. Obviously, if G is connected and has a vertex cover of size k, then it has a dominating set of size k (the same set of vertices), but the converse is not necessarily true. However, the similarity suggests that if VC in NP-complete, then DS is likely to be NP-complete as well. We will show this fact next.

As usual the proof has two parts. First we show that DS $\in$ NP. The certificate just consists of the subset V' in the dominating set. In polynomial time we can determine whether every vertex is in V' or is adjacent to a vertex in V'.

Vertex Cover to Dominating Set: Next, we show that a known NP-complete problem is reducible to dominating set. We choose vertex cover and show that VC $\leq_P$ DS. We want a polynomial time function, which given an instance of the vertex cover problem (G, k), produces an instance (G', k') of the dominating set problem, such that G has a vertex cover of size k if and only if G' has a dominating set of size k'.

How to we translate between these problems? The key difference is the condition. In VC: "every edge is incident to a vertex in V'". In DS: "every vertex is either in V' or is adjacent to a vertex in V'". Thus the translation must somehow map the notion of "incident" to "adjacent". Because incidence is a property of edges, and adjacency is a property of vertices, this suggests that the reduction function maps edges of G into vertices in G', such that an incident edge in G is mapped to an adjacent vertex in G'.

This suggests the following idea (which does not quite work). We will insert a vertex into the middle of each edge of the graph. In other words, for each edge $\{u, v\}$, we will create a new *special vertex*, called w_{uv}, and replace the edge $\{u, v\}$ with the two edges $\{u, w_{uv}\}$ and $\{v, w_{uv}\}$. The fact that u was incident to edge $\{u, v\}$ has now been replaced with the fact that u is adjacent to the corresponding vertex w_{uv}. We still need to dominate the neighbor v. To do this, we will leave the edge $\{u, v\}$ in the graph as well. Let G' be the resulting graph.

This is still not quite correct though. Define an *isolated vertex* to be one that is incident to no edges. If u is isolated it can only be dominated if it is included in the dominating set. Since it is not incident to any edges, it does not need to be in the vertex cover. Let V_I denote the isolated vertices in G, and let n_I denote the number of isolated vertices. The number of vertices to request for the dominating set will be $k' = k + n_I$. Okay, we are now ready to state the result and prove it.

Theorem: DS is NP-complete.

DS $\in$ NP: Given an instance (G, k) for DS, we guess the certificate, which consists of the k vertices that will form the dominating set. We then verify that these vertices form a dominating set, by checking that every vertex of G is either in this set or is adjacent to a vertex in this set. If so, we output "yes" and otherwise "no". (Again, if G has a dominating set of size k, one of these guesses will work, and we correctly classify G as having a dominating set of size k. Otherwise, all fail and we classify G as not having such a dominating set.)

VC $\leq_P$ DS: We want to show that given an instance of the VC problem (G, k), we can produce an equivalent instance of the DS problem in polynomial time. We create a graph G' as follows. Initially $G' = G$. For each edge $\{u, v\}$ in G we create a new vertex w_{uv} in G' and add edges $\{u, w_{uv}\}$ and $\{v, w_{uv}\}$ in G'. Let I denote the number of isolated vertices and set $k' = k + n_I$. Output (G', k'). This reduction illustrated in Fig. 2. Note that every step can be performed in polynomial time.

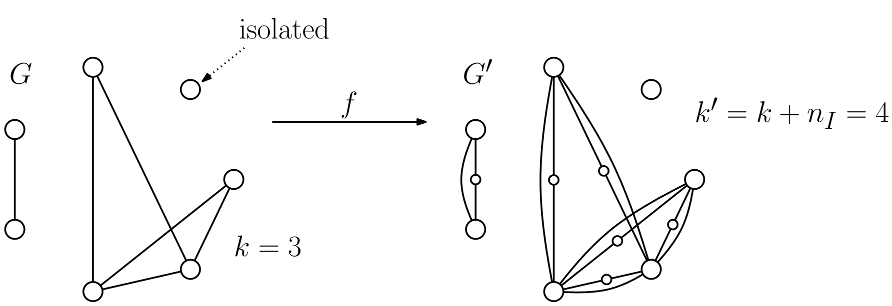

Fig. 2: Dominating set reduction with $k = 3$ and one isolated vertex.

To establish the correctness of the reduction, we need to show that G has a vertex cover of size k if and only if G' has a dominating set of size k'.

($\Rightarrow$) First we argue that if V' is a vertex cover for G, then $V'' = V' \cup V_I$ is a dominating set for G'. Observe that

$$|V''| \;=\; |V' \cup V_I| \;\leq\; k + n_I \;=\; k'.$$

Note that $|V' \cup V_I|$ might be of size less than $k + n_I$, if there are any isolated vertices in V'. If so, we can add any vertices we like to make the size equal to k'.

To see that V'' is a dominating set, first observe that all the isolated vertices are in V'' and so they are dominated. Second, each of the special vertices w_{uv} in G' corresponds to an edge $\{u, v\}$ in G implying that either u or v is in the vertex cover V'. Thus w_{uv} is dominated by the same vertex in V'' Finally, each of the nonisolated original vertices v is incident to at least one edge in G, and hence either it is in V' or else all of its neighbors are in V'. In either case, v is either in V'' or adjacent to a vertex in V''. This is shown in the top part of the following Fig. 3.

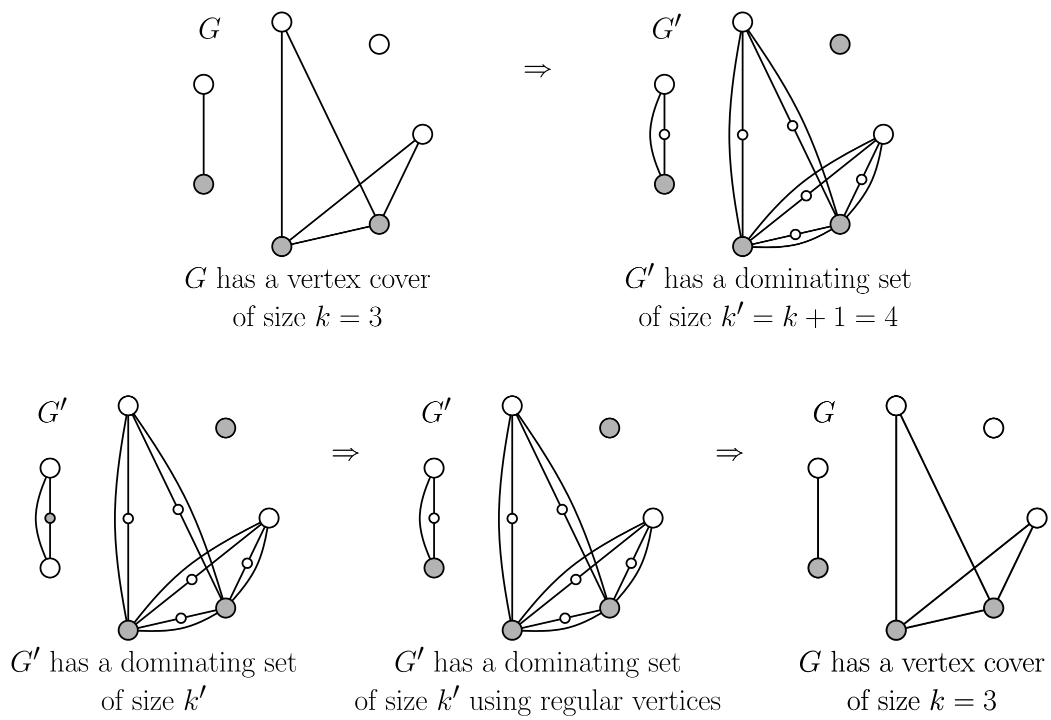

Fig. 3: Correctness of the VC to DS reduction (where $k = 3$ and $I = 1$).

($\Leftarrow$) Conversely, we claim that if G' has a dominating set V'' of size $k' = k + n_I$ then G has a vertex cover V' of size k. Note that all n_I isolated vertices of G' must be in the dominating set. First, let $V''' = V'' \setminus V_I$ be the remaining k vertices. We might try to claim something like: V''' is a vertex cover for G. But this will not necessarily work, because V''' may have vertices that are not part of the original graph G.

However, we claim that we never need to use any of the newly created special vertices in V'''. In particular, if some vertex $w_{uv} \in V'''$, then modify V''' by replacing w_{uv} with u. (We could have just as easily replaced it with v.) Observe that the vertex w_{uv} is adjacent to only u and v, so it dominates itself and these other two vertices. By using u instead, we still dominate u, v, and w_{uv} (because u has edges going to v and w_{uv}). Thus by replacing $w_{u,v}$ with u we dominate the same vertices (and potentially more). Let V' denote the resulting set after this modification. (This is shown in the lower middle part of Fig 3.)

We claim that V' is a vertex cover for G. If, to the contrary there were an edge $\{u, v\}$ of G that was not covered (neither u nor v was in V') then the special vertex w_{uv} would not be adjacent to any vertex of V'' in G', contradicting the hypothesis that V'' was a dominating set for G'.

Whew! So, this completes the proof of the correctness of the reduction, so we conclude that DS is NP-complete.