ASSIGNMENT 5 CMSC 858K (Fall 2017)

Due by 12:30 pm on Thursday, November 16. Submit PDF via ELMS (https://myelms.umd.edu).

1. *Unambiguous state discrimination.* In this problem, we will explore a task for which POVMs outperform orthogonal measurement. As in problem 4 of Assignment 1, fix some angle θ, and suppose someone flips a fair coin and, depending on the outcome, gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle$$

(but does not tell you which).

(a) [2 points] Show that if $\theta \in [0, \pi/2)$, no orthogonal measurement can distinguish between the states perfectly.

(b) [3 points] Show that there is an orthogonal measurement that *unambiguously discriminates* the states for any $\theta \in (0, \pi/2)$, meaning that if it reports an answer, it is guaranteed to be correct, but it is allowed to sometimes report that the measurement was inconclusive. What is the probability of obtaining an inconclusive result with an unambiguous discrimination procedure using orthogonal measurement?

(c) [3 points] Now suppose we try to unambiguously discriminate the states using a POVM. What is the best possible success probability? Compare your answer to that of the previous part.

(d) [3 bonus points] Describe how to implement the optimal POVM of the previous part using a unitary interaction with an ancilla followed by an orthogonal measurement.

2. *Spindles and pancakes.*

(a) [2 points] Consider a map on density matrices that sends a state with Bloch vector (x, y, z) to one with Bloch vector $(0, 0, z)$. Show that this map is a quantum operation.

(b) [3 points] Consider a map on density matrices that sends a state with Bloch vector (x, y, z) to one with Bloch vector $(x, y, 0)$. Is this map a quantum operation? Prove that your answer is correct.

3. *Effect of noise on state distinguishability.*

Continuing Problem 4 of Assignment 1 and Problem 1 above, consider the task of distinguishing $|\psi\rangle = |0\rangle$ and $|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, but now assuming we only have access to the states after they pass through a depolarizing channel.

(a) [2 points] Recall that the qubit depolarizing channel with parameter $p \in [0, 1]$ has Kraus form $\mathcal{D}_p(\rho) = (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$. Compute the action of the depolarizing channel on the states $|\psi\rangle$ and $|\phi\rangle$.

(b) [3 points] Compute the trace distance between $\mathcal{D}_p(|\psi\rangle\langle\psi|)$ and $\mathcal{D}_p(|\phi\rangle\langle\phi|)$.

(c) [1 point] Discuss how the depolarizing channel affects the probability of distinguishing the states.

(d) [1 point] If $p > 0$, is it possible to find a POVM that *unambiguously* distinguishes the states $\mathcal{D}_p(|\psi\rangle\langle\psi|)$ and $\mathcal{D}_p(|\phi\rangle\langle\phi|)$ with nonzero probability (in the sense of Problem 1 above)?

4. *Three-bit repetition codes under damping channels.*

   (a) [3 points] Consider encoding a general qubit state $\alpha|0\rangle + \beta|1\rangle$ into the 3-bit repetition code as $\alpha|000\rangle + \beta|111\rangle$. Recall that the amplitude damping channel $\mathcal{A}_\gamma$ acts as $\mathcal{A}_\gamma(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$ with Kraus operators $E_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{smallmatrix}\right)$ and $E_1 = \left(\begin{smallmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{smallmatrix}\right)$. Compute the effect of $\mathcal{A}_\gamma$ on the first qubit of the encoded state, and find the probabilities for each possible syndrome measurement.

   (b) [1 point] What is the state of the system after error correction?

   (c) [1 point] What is the fidelity between the input and output states?

   (d) [3 points] Recall the phase damping channel $\mathcal{P}_\lambda$ acts as $\mathcal{P}_\lambda(\rho) = F_0\rho F_0^\dagger + F_1\rho F_1^\dagger$ with Kraus operators $F_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{smallmatrix}\right)$ and $F_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{smallmatrix}\right)$. Now consider encoding the qubit into the 3-bit phase flip code as $\alpha|+++\rangle + \beta|---\rangle$. Compute the effect of the phase damping channel $\mathcal{P}_\lambda$ on the first qubit, and find the probabilities for each possible syndrome measurement.

   (e) [1 point] What is the state of the system after error correction?

   (f) [1 point] What is the fidelity between the input and output states?

5. *The five-qubit code.* Consider a quantum error correcting code that encodes one logical qubit into five physical qubits, with the logical basis states

$$
\begin{aligned}
|0_L\rangle = \tfrac{1}{4}(&|00000\rangle \\
&+|10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\
&-|11000\rangle - |01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle \\
&-|01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle - |11110\rangle) \\
|1_L\rangle = \tfrac{1}{4}(&|11111\rangle \\
&+|01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\
&-|00111\rangle - |10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle \\
&-|10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle - |00001\rangle).
\end{aligned}
$$

   (a) [4 points] Show that $|0_L\rangle$ and $|1_L\rangle$ are simultaneous eigenstates (with eigenvalue +1) of the operators given in equation 10.5.18 of KLM. (Hint: You can show this without explicitly checking every case.)

   (b) [4 points] Show that this code can correct an X or Z error acting on any of the five qubits. You should explain how the different possible errors would be reflected by a measurement of the error syndrome.

   (c) [1 point] Explain why this means that the code can correct any single-qubit error.

   (d) [2 points] Find logical Pauli operators X_L and Z_L such that $X_L|0_L\rangle = |1_L\rangle$, $X_L|1_L\rangle = |0_L\rangle$, $Z_L|0_L\rangle = |0_L\rangle$, and $Z_L|1_L\rangle = -|1_L\rangle$.

   (e) [3 bonus points] Give a quantum circuit that computes the syndrome of the five-qubit code.

6. *An error-detecting code.* Suppose we encode one logical qubit into four physical qubits, using the logical basis states

$$|0_L\rangle = \frac{1}{2}(|0000\rangle + |1100\rangle + |0011\rangle + |1111\rangle)$$
$$|1_L\rangle = \frac{1}{2}(|0110\rangle + |1010\rangle + |0101\rangle + |1001\rangle).$$

(a) [3 points] Show that $|0_L\rangle$ and $|1_L\rangle$ are simultaneous eigenstates (with eigenvalue $+1$) of the operators $X \otimes X \otimes I \otimes I$, $I \otimes I \otimes X \otimes X$, and $Z \otimes Z \otimes Z \otimes Z$.

(b) [3 points] Find two distinct single-qubit Pauli errors with the same syndrome. This shows that the code fails to correct an arbitrary single-qubit error.

(c) [3 points] Show that any single-qubit Pauli error anticommutes with at least one of the three operators from part (a). (We say that A and B anticommute if $AB = -BA$.) Explain why this means that the code can detect whether a single-qubit error occurred, even though (by the previous part) it cannot determine how to correct the error.