Notes by Laura Bright.

50 Linear Programming: The Use of Duality

The Primal-Dual Method is a useful tool for solving combinatorial optimization problems. In this lecture we first study Duality and see how it can be used to design an algorithm to solve the Assignment problem.
Dual Problem Motivation

The primal problem is defined as follows:

Maximize

$$\sum_{j=1}^{n} c_j x_j$$

subject to:

$$\forall i \in (1, \ldots, m) \sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$x_j \geq 0$$

To formulate the dual, for each constraint introduce a new variable y_i, multiply constraint i by that variable, and add all the constraints.

$$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \leq \sum_{i=1}^{m} b_i y_i$$

Consider the coefficient of each x_j.

$$\sum_{i=1}^{m} a_{ij} y_i$$

If this coefficient is at least c_j

$$c_j \leq \sum_{i=1}^{m} a_{ij} y_i$$

then we can use $\sum_{i=1}^{m} b_i y_i$ as an upper bound on the maximum value of the primal LP.

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j \leq \sum_{i=1}^{m} b_i y_i$$

We want to derive the best upper bound for the dual so we wish to minimize this quantity.
Dual:

Minimize

$$\sum_{i=1}^{m} b_i y_i$$

subject to:

$$\forall j \in (1, \ldots, n) \sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

$$y_i \geq 0$$

The constraint that each y_i be positive is to preserve the inequalities of the primal problem. The constraints of the primal problem could also be equality constraints, and in this case we can drop the requirement that the y_i values be positive. It is worth noting that if you take the dual of the dual problem, you will get back the original primal problem.

We saw in the previous lecture that using the simplex method, you can find the optimal solution to a primal problem if one exists. While the simplex certifies its own optimality, the dual solution can be used to certify the optimality of any given primal solution. The *Strong Duality Theorem* proves this.

Theorem 50.1 (Strong Duality Theorem)
If the primal LP has an optimal solution x^*, then the dual has an optimal solution y^* such that:

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

Proof:

To prove the theorem, we only need to find a (feasible) solution y^* that satisfies the constraints of the Dual LP, and satisfies the above equation with equality. We solve the primal program by the simplex method, and introduce m slack variables in the process.

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad (i = 1, \ldots, m)$$

Assume that when the simplex algorithm terminates, the equation defining z reads as:

$$z = z^* + \sum_{k=1}^{n+m} \overline{c}_k x_k.$$

Since we have reached optimality, we know that each $\overline{c}_k$ is a nonpositive number (in fact, it is 0 for each basic variable). In addition z^* is the value of the objective function at optimality, hence $z^* = \sum_{j=1}^{n} c_j x_j^*$. To produce y^* we pull a rabbit out of a hat ! Define $y_i^* = -\overline{c}_{n+i} \quad (i = 1, \ldots, m)$.

To show that y^* is an optimal dual feasible solution, we first show that it is feasible for the Dual LP, and then establish the strong duality condition.

From the equation for z we have:

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{k=1}^{n} \overline{c}_k x_k - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j).$$

Rewriting it, we get

$$\sum_{j=1}^{n} c_j x_j = (z^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (\overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*) x_j.$$

Since this holds for all values of x_i, we obtain:

$$z^* = \sum_{i=1}^{m} b_i y_i^*$$

(this *establishes the equality*) and

$$c_j = \overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \quad (j = 1, \ldots, n).$$

Since $\overline{c}_k \leq 0$, we have

$$y_i^* \geq 0 \quad (i = 1, \ldots, m).$$

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \quad (j = 1, \ldots, n)$$

This *establishes the feasibility* of y^*.

□

We have shown that the dual solution can be used to verify the optimality of a primal solution. Now we will show how this is done using an example. F irst we introduce *Complementary Slackness Conditions.*

Complementary Slackness Conditions:

Theorem 50.2 *Necessary and Sufficient conditions for x^* and y^* to be optimal solutions to the primal and dual are as follows.*

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ or } x_j^* = 0 \text{ (or both) for } j = 1, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \text{ or } y_i^* = 0 \text{ (or both) for } i = 1, \ldots, m$$

In other words, if a variable is non-zero then the corresponding equation in the dual is met with equality, and vice versa.

Proof:

We know that

$$c_j x_j^* \leq (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^* \qquad (j = 1, \ldots, n)$$

$$(\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* \leq b_i y_i^* \qquad (i = 1, \ldots, m)$$

We know that at optimality, the equations are met with equality. Thus for any value of j, either $x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$. Similarly, for any value of i, either $y_i^* = 0$ or $\sum_{j=1}^{n} a_{ij} x_j^* = b_i$. □

The following example illustrates how complementary slackness conditions can be used to certify the optimality of a given solution.

Consider the primal problem:

Maximize

$$18x_1 - 7x_2 + 12x_3 + 5x_4 + 8x_6$$

subject to:

$$2x_1 - 6x_2 + 2x_3 + 7x_4 + 3x_5 + 8x_6 \leq 1$$

$$-3x_1 - x_2 + 4x_3 - 3x_4 + x_5 + 2x_6 \leq -2$$

$$8x_1 - 3x_2 + 5x_3 - 2x_4 + 2x_6 \leq 4$$

$$4x_1 + 8x_3 + 7x_4 - x_5 + 3x_6 \leq 1$$

$$5x_1 + 2x_2 - 3x_3 + 6x_4 - 2x_5 - x_6 \leq 5$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

We claim that the solution

$$x_1^* = 2, x_2^* = 4, x_3^* = 0, x_4^* = 0, x_5^* = 7, x_6^* = 0$$

is an optimal solution to the problem.

According to the first complementary slackness condition, for every strictly positive x value, the corresponding dual constraint should be met with equality. Since the values x_1, x_2, and x_5 are positive, the first, second, and fifth dual constraints must be met with equality. Similarly, from the second complementary slackness condition, we know that for every strictly positive y value, the corresponding primal constraint should be met with equality. By substituting the given x values in the primal equations, we see that the second and fifth equations are not met with equality. Therefore y_2 and y_5 must be equal to 0. The complementary slackness conditions therefore give us the following:

$$2y_1^* - 3y_2^* + 8y_3^* + 4y_4^* + 5y_5^* = 18$$

$$-6y_1^* - y_2^* - 3y_3^* + 2y_5^* = -7$$

$$3y_1^* + y_2^* - y_4^* - 2y_5^* = 0$$

$$y_2^* = 0$$

$$y_5^* = 0$$

Solving this system of equations gives us $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$. By formulating the dual and substituting these y values, it is easy to verify that this solution satisfies the constraints of the dual, and therefore our primal solution must be optimal.

Note that this method of verifying solutions only works if the system of equations has a unique solution. If it does not, it means that the dual is unbounded, and the primal problem is therefore infeasible. Similarly, if the primal problem is unbounded, the dual is infeasible.

Assignment Problem

We can now revisit the weighted matching problem we studied earlier in the semester to see how the solution to this problem can be derived using the primal-dual method. The Assignment Problem is, given a complete bipartite graph with a weight w_e assigned to each edge, to find a maximum weighted perfect matching. Stated as a linear program, the problem is as follows:

Maximize

$$\sum w_e x_e \quad 1 \geq x_e \geq 0$$

subject to:

$$\forall u \in U \sum_{e=(u,*)} x_e = 1$$

$$\forall v \in V \sum_{e=(*,v)} x_e = 1$$

The dual to this problem is:

Minimize

$$\sum_{u \in U} Y_u + \sum_{v \in V} Y_v$$

subject to:

$$Y_u + Y_v \geq w(u,v) \quad \forall edges(u,v)$$

Since all the constraints of the primal are met with equality, the second complementary slackness condition is automatically satisfied. By the definition of complementary slackness, if we have a feasible primal solution that satisfies the first complementary slackness condition, we are done.

In any feasible solution to the primal, every vertex has some edge e such that $x_e > 0$. By the first complementary slackness condition, if $x_j^* > 0$ then the corresponding dual constraint must be met with equality. When we have a feasible primal solution such that $Y_u + Y_v = w(u, v)$ for all edges $e = (u, v)$ such that $x_e > 0$, we have an optimal solution to both problems. This suggests that only the edges in the "equality subgraph" (edges for which the dual constraints are met with equality) should be considered when we want to add edges to the matching. Also notice that the number of unmatched vertices is our "measure of infeasibility" – as this decreases, we approach feasibility.

The labeling function l on the vertices that we used earlier is nothing but the dual variables.

$$l(u_i) + l(v_j) \geq w(u_i, v_j)$$

Start with a feasible labeling and compute the equality subgraph G_l which includes all the vertices of the original graph G but only edges (x_i, y_j) which have weights such that $w(x_i, y_j) = l(x_i) + l(y_j)$.

If G_l is a perfect matching, we have an optimal solution. Otherwise, we revise the labels to improve the quality of the matching.

The algorithm finds a maximum matching in the equality subgraph. We increase the labels of some vertices and decrease the labels of others. The total sum of the labels is dropping, so we are decreasing our dual solution. If we increase the size of the matching, we are one step closer to the optimal solution. When an optimal labeling is found, we have an optimal solution to the dual, and we therefore have a maximum weighted matching in G.