Quaternions

Why quaternions?

In computer graphics, we often have to specify rotation, or orientation, of an object in relation to the world coordinate system. For aerodynamic model, we need to calculate the rotational velocities, etc relative to the fixed body coordinate system. There are 3 main approaches for specifying orientation and converting the angular velocities to the orientation of a moving object in the world coordinates: Euler Method, homogeneous matrices, and quaternions. Each has its own particular merits and disadvantages.

The most popular and intuitive method is probably the Euler's Method, which uses a sequence of three angles to describe the orientation of a moving object; while the homogeneous matrix is a convenient way to specify the transformation of coordinate systems and it eliminates the intermediate process to compute some transcendental functions required by the Euler's Method in aerodynamics calculations. So, why quaternions?

Quaternions require 4 parameters (defined below) vs. 3 in the Euler's Method. But, quaternions don't have the singularity problem at $\theta = 90$ or -90 degrees. From user interface's point of view, quaternions don't require the user to enter all elements in the homogeneous matrices. Unit quaternions (having norm = 1) capture all the geometry, topology, and group structure of 3-dimensional rotations in the simplest and compact way possible, though they have problem in defining the unique Euler's angles, if such information is needed. Nevertheless, quaternions have become a powerful, simpler, cheaper, and better behaved treatment of rotations in computer graphics.

Basics and Definitions

Quaternions can be defined in several different but equivalent ways. Quaternion can be described as an algebraic quantity $q = ix + jy + kz + w$, or as a point linearly transformed to 4-space (x, y, z, w), or as a 3-vector with a scalar [**v**, **w**]. But, it always comprises four parameters, 1 of which is the scalar part and the rest 3 parameters is the vector portion.

FORMS:

$$q = [v, w] = [(x, y, z);w] = [x, y, z, w];v \in R^3;x, y, z, w \in R$$

$$q = ix + jy + kz + w;x, y, z, w \in R;i^2 = j^2 = k^2 = -1$$

ADDITION:

$$q + q' = [v, w] + [v', w'] = [v + v', w + w']$$

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MULTIPLICATION:

$$qq' = [v, w]\ [v', w'] = (ix + jy + kz + w)\ (ix' + jy' + kz' + w')$$

$$qq' = (ww' - (xx' + yy' + zz')) + wv' + w'v + v \times v' = [\ (wv' + w'v + v \times v'),\ ww' - v \bullet v']$$

Note: "x" and "•" denote for vector cross product and dot product.

CONJUGATE:

$$q^* = [v, w]^* = [-v, w]$$

NORM, || q || is defined as:

$$N(qq') = qq^* = q^*q = w^2 + v \bullet v = w^2 + x^2 + y^2 + w^2 = \|q\|^2$$

INVERSE:

$$q^{-1} = q^*/N(q) = q^*/\|q\|^2$$

UNIT QUATERNION is a quaternion q such that its N(q) = 1. If both q and q' are unit quaternions,

$$N(qq') = 1$$

$$q^{-1} = q^*$$

$$q = [\hat{v}\sin\Omega, \cos\Omega]\ ; \exists\hat{v}$$

All real numbers *s* can be identified by quaternions **q** = [0, *s*] and all vectors **v** = (x,y,z) can be represented by quaternion **q** = [**v**, 0]. Unit quaternions provide an efficient way to represent rotations. The orientation of a rigid body can be described as a rotation about an axis **u** by rotation angle θ. This becomes the basis of using quaternions for specifying rotation.

Rotational Operators

As mentioned above, rotation of an angle θ about some unit axis **u** can be specified by a unit quaternion. Note that this choice is not unique, since rotation by θ about **u** is the same as rotation by -θ about -**u**. (By the way, it's pretty easily to derive the corresponding 3x3 rotation matrix from θ and **u**. We'll state the relationship between the 3x3 rotation matrix and quaternions later.)

The quaternion **q** corresponding to this rotation by an angle θ about unit axis **u** is given by

q = [sin(θ/2) u, cos(θ/2)],

That is, the scalar part is $\cos(\theta/2)$, and the vector part is simply the vector **u** scaled by $\sin(\theta/2)$. This quaternion will always have unit magnitude. What does it mean to say that **q** corresponds to this rotation? We rotate vectors with quaterions by the following formula:

$$\mathbf{v}' = \mathbf{q} * \mathbf{v} * \mathbf{q}^{-1}$$

where **v** is the original vector and **v'** is the rotated one. The symbol * stands for quaternion multiplication (**v** can be turned into a quaternion just by setting the scalar part to 0). If v = [**0**, s], then a scalar after quaternion transformation is still itself by commutivity, i.e. $\mathbf{v} = \mathbf{q}*\mathbf{v}*\mathbf{q}^{-1} = \mathbf{s}*\mathbf{q}*\mathbf{q}^{-1} = \mathbf{s}$.

EXAMPLE: What quaternion corresponds to rotation by 90 degrees about the z axis?

In this case, the angle θ is $\pi/2$ and the unit axis is (**0,0,1**). So our quaternion is:

$$\mathbf{q} = [\sin(\pi/4)\ (0,0,1);\ \cos(\pi/4)] = [\ 0,\ 0,\ 0.707,\ 0.707]$$

Note that q is a unit quaternion, as we expect. Since it is a unit quaternion, it's inverse is easy to find --- just reverse the vector part:: $\mathbf{q}^{-1} = [0,\ 0,\ -0.707,\ 0.707]$ Now if we take any vector v, and make it into a quaternion by setting the scalar part to 0, and then compute $\mathbf{v}' = \mathbf{q} * \mathbf{v} * \mathbf{q}^{-1}$, then v' will be v rotated by 90 degrees about the z axis.

Given a unit vector $\mathbf{u} = (\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z})^T$, and an angle θ, the rotation matrix corresponding to rotation about **u** by θ is given by

$$R = \begin{bmatrix} u_x u_x V_\theta + \cos\theta & u_y u_x V_\theta - u_z \sin\theta & u_z u_x V_\theta + u_y \sin\theta \\ u_x u_y V_\theta + u_z \sin\theta & u_y u_y V_\theta + \cos\theta & u_z u_y V_\theta - u_x \sin\theta \\ u_x u_z V_\theta - u_y \sin\theta & u_y u_z V_\theta + u_x \sin\theta & u_z u_z V_\theta + \cos\theta \end{bmatrix} \qquad (*)$$

where $V_\theta = 1 - \cos\theta$. The corresponding unit quaternion is given by $\mathbf{q} = [\ (\sin\theta/2)\ \mathbf{u},\ \cos\theta/2] = [\ \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\ ]$. We can invert the relation to express **u** and trig functions of θ in terms of quaternion parameters (assuming $0 <= \theta <= 2\pi$) by the following equations:

$$\cos\theta = 2w^2 - 1$$

$$\sin\theta = 2w\sqrt{1 - w^2}$$

$$V_\theta = 2\left(1 - w^2\right)$$

$$u = \frac{1}{\sqrt{1 - w^2}}(x, y, z)^T$$

Substituting the above equations into the rotation matrix R in (*) gives us a formula for a rotation matrix in terms of the parameters of the corresponding quaternion:

$$R = \begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & w^2 - x^2 + y^2 - z^2 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & w^2 - x^2 - y^2 + z^2 \end{bmatrix}$$

The detailed theorem and derivation of this conversion is given in both the SIGGRAPH'93 course notes and the paper titled, "**Flight Simulation Dynamic Modeling Using Quaternions**" by Cooke/Zyda/Pratt/McGhee as well. More details on rotational derivatives will be described later in flight dynamic modeling section.

REVIEW:

$$\vec{v} \bullet \vec{u} = |\vec{v}||\vec{u}|\cos\theta$$

$$\vec{r} = \vec{v} \times \vec{u} \Leftrightarrow |\vec{r}| = |\vec{v}||\vec{u}|\sin\theta$$

$$[x_1, y_1, z_1] \times [x_2, y_2, z_2] = [y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - x_2y_1]$$

1 Quaternion basics

A quaternion comprises four parameters:

$$q = [q_s; q_x, q_y, q_z] \tag{1}$$

where q_s is the scalar portion, and $(q_x, q_y, q_z)^T$ is the vector portion. Sometimes we abbreviate and write

$$q = [s, \vec{v}]. \tag{2}$$

A multiplication operation on quaternions is defined as

$$[s_1, \vec{v}_1] * [s_2, \vec{v}_2] = [s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2, s_1\vec{v}_2 + s_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]. \tag{3}$$

For $q = [s, \vec{v}]$, we also have

$$\bar{q} = [s, -\vec{v}] \tag{4}$$

$$||q|| = \sqrt{q * \bar{q}} = \sqrt{s^2 + ||\vec{v}||^2} \tag{5}$$

$$q^{-1} = \frac{1}{q} = \frac{\bar{q}}{||q||^2} \tag{6}$$

$$\tag{7}$$

Note that if q is a unit quaternion, $q^{-1} = \bar{q}$. Unit quaternions correspond to rotations (elements of $SO(3)$). Let $\vec{v}$ be a vector to be rotated by quaternion q. We have the following formula for the roated vector $\vec{v}_r$:

$$\vec{v}_r = q * \vec{v} * q^{-1}. \tag{8}$$

In the above equation, the vectors $\vec{v}$ and $\vec{v}_r$ are "casted" to a quaternion by setting the scalar part of the quaternion equal to 0. For example, the vector $\vec{v}$ is identified with the quaternion $[0, \vec{v}]$.

2 Rotational operators: R and q

Given a unit vector $\hat{k} = (k_x, k_y, k_z)^T$, and an angle θ, The rotation matrix corresponding to rotation about $\hat{k}$ by θ is given by

$$R = \begin{bmatrix} k_x k_x V_\theta + C_\theta & k_y k_x V_\theta - k_z S_\theta & k_z k_x V_\theta + k_y S_\theta \\ k_x k_y V_\theta + k_z S_\theta & k_y k_y V_\theta + C_\theta & k_z k_y V_\theta - k_x S_\theta \\ k_x k_z V_\theta - k_y S_\theta & k_y k_z V_\theta + k_x S_\theta & k_z k_z V_\theta + C_\theta \end{bmatrix}, \tag{9}$$

where $C_\theta = \cos\theta$, $S_\theta = \sin\theta$, and $V_\theta = \text{vers}\,\theta = 1 - \cos\theta$. The corresponding unit quaternion is given by

$$q = [\cos\frac{\theta}{2}, (\sin\frac{\theta}{2})\hat{k}]. \tag{10}$$

This gives the quaternion parameters $[q_s; q_x, q_y, q_z]$ in terms of $\hat{k}$ and θ. We can invert the realation to express $\hat{k}$ and trig functions of θ in terms of quaternion parameters (assuming $0 \leq \theta \leq 2\pi$):

$$\cos\theta = 2{q_s}^2 - 1 \tag{11}$$

$$\sin\theta = 2q_s\sqrt{1 - {q_s}^2} \tag{12}$$

$$vers\ \theta = 2(1 - {q_s}^2) \tag{13}$$

$$\hat{k} = \frac{1}{\sqrt{1 - {q_s}^2}}(q_x, q_y, q_z)^T \tag{14}$$

Substituting the above equations into (9) gives us a fromula for a rotation matrix in terms of the parameters of the corresponding quaternion:

$$R = 2\begin{bmatrix} {q_x}^2 + {q_s}^2 - \frac{1}{2} & q_y q_x - q_z q_s & q_z q_x + q_y q_s \\ q_x q_y + q_z q_s & {q_y}^2 + {q_s}^2 - \frac{1}{2} & q_z q_y - q_x q_s \\ q_x q_z - q_y q_s & q_y q_z + q_x q_s & {q_z}^2 + {q_s}^2 - \frac{1}{2} \end{bmatrix}, \tag{15}$$

3 Rotational derivatives: $\dot{R}$, $\vec{\omega}$, and $\dot{q}$

Differentiating (15) gives us $\dot{R}$ in terms of q and $\dot{q}$:

$$\dot{R} = 2\begin{bmatrix} 2(q_x\dot{q}_x + q_s\dot{q}_s) & \dot{q}_y q_x + q_y\dot{q}_x - \dot{q}_z q_s - q_z\dot{q}_s & \dot{q}_z q_x + q_z\dot{q}_x - \dot{q}_y q_s - q_y\dot{q}_s \\ \dot{q}_x q_y + q_x\dot{q}_y - \dot{q}_z q_s - q_z\dot{q}_s & 2(q_y\dot{q}_y + q_s\dot{q}_s) & \dot{q}_z q_y + q_z\dot{q}_y - \dot{q}_x q_s - q_x\dot{q}_s \\ \dot{q}_x q_z + q_x\dot{q}_z - \dot{q}_y q_s - q_y\dot{q}_s & \dot{q}_y q_z + q_y\dot{q}_z - \dot{q}_x q_s - q_x\dot{q}_s & 2(q_z\dot{q}_z + q_s\dot{q}_s) \end{bmatrix}, \tag{16}$$

Now for any rotation matrix R which is changing with time according to $\vec{\omega}$ we have the relation

$$\dot{R} = (\omega\times)R, \tag{17}$$

where $\omega\times$ is the skew symmetric matrix such that $(\omega\times)\vec{v} = \vec{\omega}\times\vec{v}$ for all $\vec{v}$. That is,

$$\omega\times = \begin{bmatrix} 0 & -\omega_z & +\omega_y \\ +\omega_z & 0 & -\omega_x \\ -\omega_y & +\omega_x & 0 \end{bmatrix} \tag{18}$$

¿From (17) we have

$$\omega\times = \dot{R}R^{-1} = \dot{R}R^T. \tag{19}$$

Substituting (16) and (15) into the above equation gives us an expression for $\omega\times$ soley in terms of q and $\dot{q}$. [After performing the matrix multiplication on the right hand side of (19), we are left with an extremely hairy matrix. However, it is much nicer than it initially appears. Using the equation ${q_s}^2 + {q_x}^2 + {q_y}^2 + {q_z}^2 = 1$ and its derivative, $q_s\dot{q}_s + q_x\dot{q}_s + q_y\dot{q}_y + q_z\dot{q}_z = 0$, many terms drop out, and we are left with a very simple matrix.] Because of the trivial mapping between the vector $\vec{\omega}$ and the matrix $\omega\times$, once we have the an expression for $\omega\times$, it is a simple matter of picking out the proper matrix elements to obtain expressions for the three components of $\vec{\omega}$ in terms of q and $\dot{q}$. After doing this, we find

$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 2\begin{bmatrix} -q_x & +q_s & -q_z & +q_y \\ -q_y & +q_z & +q_s & -q_x \\ -q_z & -q_y & +q_x & +q_s \end{bmatrix}\begin{bmatrix} \dot{q}_s \\ \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix}. \tag{20}$$

It seems we are mapping a four dimensional space to a three dimensional space, but actually because of the constraint that q be of unit magnitude, there are really only three degrees of freedom in $\dot{q}$. The constraint can be written $q_x\dot{q}_x + q_y\dot{q}_y + q_z\dot{q}_z + q_s\dot{q}_s = 0$. Incorporating this into the above equation gives us a 4 x 4 transformation matrix:

$$\begin{bmatrix} \vec{\omega} \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ 0 \end{bmatrix} = 2\begin{bmatrix} -q_x & +q_s & -q_z & +q_y \\ -q_y & +q_z & +q_s & -q_x \\ -q_z & -q_y & +q_x & +q_s \\ +q_s & +q_x & +q_y & +q_z \end{bmatrix}\begin{bmatrix} \dot{q}_s \\ \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix}. \tag{21}$$

The matrix of (21) is always invertible, and so we can also express $\dot{q}$ in terms of $\vec{\omega}$:

$$\dot{q} = \begin{bmatrix} \dot{q}_s \\ \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -q_x & -q_y & -q_z & +q_s \\ +q_s & +q_z & -q_y & +q_x \\ -q_z & +q_s & +q_x & +q_y \\ +q_y & -q_x & +q_s & +q_z \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -q_x & -q_y & -q_z \\ +q_s & +q_z & -q_y \\ -q_z & +q_s & +q_x \\ +q_y & -q_x & +q_s \end{bmatrix}\vec{\omega}. \tag{22}$$