Range Trees I

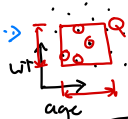
Can we do better?

Range Trees:
- Space is $O(n \log^{d-1} n)$
- Query time:
  - Counting: $O(\log^d n)$
  - Reporting: $O(k + \log^d n)$

→ In $\mathbb{R}^2$: $\log^2 n$ much better than $\sqrt{n}$ for large n

→ Range trees are more limited

Layering: Combing search structures
- Suppose you want to answer a composite query w. multiple criteria:
- Medical data: Count subjects
  - Age range: $a_{lo} \le age \le a_{hi}$
  - Weight range: $w_{lo} \le weight \le w_{hi}$



- Design a data structure for each criterion individually
- Layer these structures together to answer full query

→ Multi-Layer Data Structures

Recap:
- kd-Tree: General-purpose data structure for pts in $\mathbb{R}^d$
- Orthogonal range query: Count/report pts in axis-aligned rect.

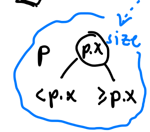
- kd-Tree: Counting: $O(\sqrt{n})$ time
  - Report: $O(k + \sqrt{n})$ time

No. of pts reported

1-Dim Range Tree:

→ Count: 5
Rept: {g, a, c, e}

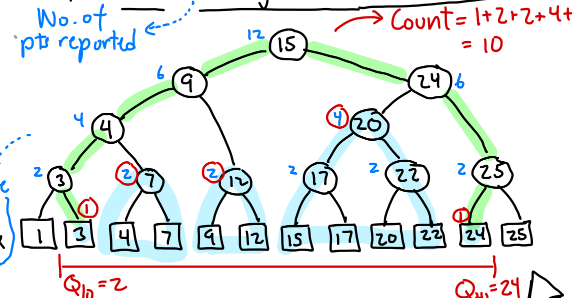
Approach:
- Balanced BST (eg. AVL, RB, ..)
- Assume extended tree
- Each node p stores no. of entrie in subtree: p.size



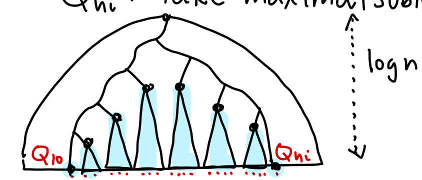
Call this a 1-Dim Range Tree:

Claim: A 1-Dim range tree with n pts has space $O(n)$ and answers 1-D range count/rept queries in time $O(\log n)$ (or $O(k + \log n)$)

No. of pts reported



Canonical Subsets:
- Goal: Express answer as disjoint union of subsets
- Method: Search for Q_{lo} + Q_{hi} + take maximal subtrees

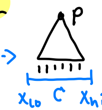


Range Trees II

More details:

Given a 1-D range tree T:

- Let $Q = [Q_{lo}, Q_{hi}]$ be query interval
- For each node p, define interval cell $C = [x_0, x_1]$ s.t. all pts of p's subtree lie in C
- Root cell: $C_0 = [-\infty, +\infty]$



Recursive helper:

int range1Dx(Node p, Intv $Q = [Q_{lo}, Q_{hi}]$, Intv $C = [x_0, x_1]$)

initial call: range1Dx(root, Q, C_0)

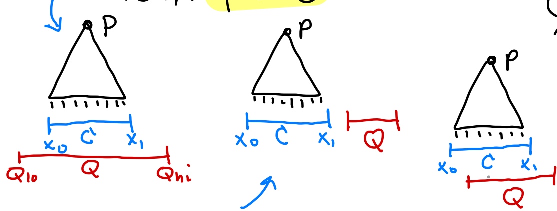
Cases:

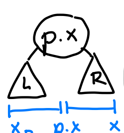
p is external:

- if $p.pt.x \in Q \to 1$ else $\to 0$

p is internal:

- $C \subseteq Q \Rightarrow$ all of p's pts lie within query
  → return p.size
- C is disjoint from Q ⇒ none of p's pts lie in Q
  → return 0
- Else partial overlap
  → Recurse on p's children + trim the cell






```
int range1Dx(Node p,
    Intv Q, Intv C=[x0,x1]) {
  if (p is external)
    return p.pt.x ∈ Q   (T → 1, F → 0)
  else if (C ⊆ Q) return p.size
  else if (Q + C disjoint) return 0
  else return:
      range1Dx(p.left, Q, [x0, p.x])
    + range1Dx(p.right, Q, [p.x, x1])
```

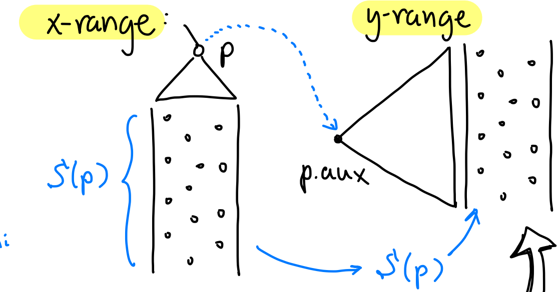
Analysis:

Lemma: Given a 1-D range tree with n pts, given any interval Q, can compute $O(\log n)$ subtrees whose union is answer to query.

Thm: Given 1-D range tree... can answer range queries in time $O(\log n)$ (+k to report)

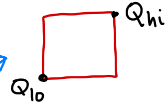
2-D Range Searching:

- "Layer" a range tree for x with range tree for y
- For each node $p \in$ 1D-x tree, let $S(p)$ = set of pts in p's subtree
- Def: p.aux: A 1D-y tree for $S(p)$



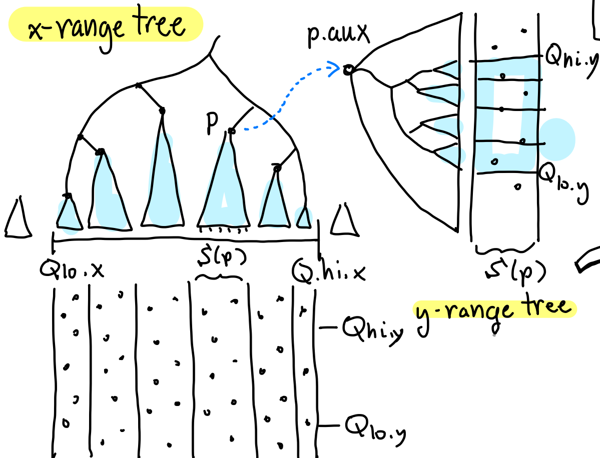
Range Trees III

Answering Queries?



Given query range $Q = [Q_{lo}.x, Q_{hi}.x] \times [Q_{lo}.y, Q_{hi}.y]$

- Run range1Dx to find all subtrees that contribute
- For each such node p, run range1Dy on p.aux
- Return sum of all result



Intuition: The x-layer finds subtrees p contained in x-range + each aux tree filters based on y.

2D Range Tree:

- Construct 1D range tree based on x coords for all pts
- For each node p:
  - Let $S(p)$ be pts of p's tree
  - Build 1D range tree for $S(p)$ based on $y \rightarrow$ p.aux
- Final structure is union of x-tree + (n-1) y-trees

Higher Dimensions?

- In d-dim space, we create d-layers
- Each recurses one dim lower until we reach 1-d search
- Time is the product: $\underbrace{\log n \cdot \log n \cdots \log n}_{d} = O(\log^d n)$

Analysis: The 1D x search takes of $O(\log n)$ time + generates $O(\log n)$ calls to 1D y search

$\Rightarrow$ Total: $O(\log n \cdot \log n) = O(\log^2 n)$

```
int range2D(Node p, Rect Q, Intv C=[x0,x1]){
    if (p is external) return p.pt ∈ Q ? (T→1, F→0)
    else if (Q.x contains C){        // C ⊆ Q's x-projection
        [y0,y1] = [-∞,+∞]             // init y-cell
        return range1Dy(p.aux, Q, [y0,y1])
    } else if (Q.x is disjoint of C) return 0
    else                              // partial x-overlap
        return range2D(p.left, Q, [x0, p.x])
             + range2D(p.right, Q, [p.x, x1])
}
```

Analysis:

Invoked $O(\log n)$ times - once per maximal subtree

Invoked $O(\log n)$ times - once for each ancestor of max subtree