

## CMSC 754: Lecture 6

### Halfplane Intersection and Point-Line Duality

**Reading:** Chapter 4 in the 4M's, with some elements from Sections 8.2 and 11.4.

**Halfplane Intersection:** Today we begin studying another fundamental topic in geometric computing and convexity. Recall that any line in the plane splits the plane into two regions, one lying on either side of the line. Each such region is called a *halfplane*. We say that a halfplane is either *closed* or *open* depending, respectively, on whether or not it contains the line. Unless otherwise stated, we will assume that halfplanes are closed.

In the *halfplane intersection problem*, we are given a collection of  $n$  halfplanes  $H = \{h_1, \dots, h_n\}$ , and the objective is to compute their intersection. It is easy to see that the intersection of halfspaces is a convex polygon (see Fig. 1(a)), but this polygon may be unbounded (see Fig. 1(b)) or even empty (see Fig. 1(c)).

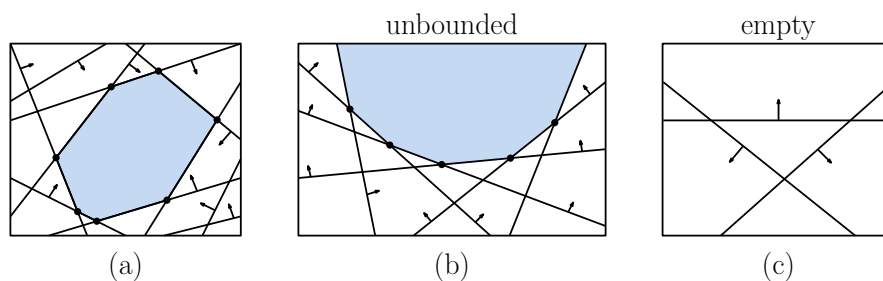


Fig. 1: Halfplane intersection.

Clearly, the number of sides of the resulting polygon is at most  $n$ , but may be smaller since some halfspaces may not contribute to the final shape.

**Halfspace Intersection:** In  $d$ -dimensional space the corresponding notion is a *halfspace*, which is the set of points lying to one side of a  $(d - 1)$ -dimensional hyperplane. The intersection of halfspaces is a *convex polytope*. The resulting polytope will have at most  $n$  facets (at most one per halfspace), but (surprisingly) the overall complexity can be much higher.

A famous result, called *McMullen's Upper-Bound Theorem* states that a polytope with  $n$  facets in dimension  $d$  can have up to  $O(n^{\lfloor d/2 \rfloor})$  vertices. (In dimensions 2 and 3, this is linear in the number of halfspaces, but even in dimension 4 the number of vertices can jump to  $O(n^2)$ .) Obtaining such a high number of vertices takes some care, but the bound is tight. There is a famous class of polytopes, called the *cyclic polytopes*, that achieve this bound. Symmetrically, the convex hull of  $n$  points in dimension  $d$  defines a convex polytope that can have  $O(n^{\lfloor d/2 \rfloor})$  facets, and this bound is also tight.

**Representing Lines and Hyperplanes:** (Digression) While we will usually treat geometric objects rather abstractly, it may be useful to explore a bit regarding how lines, halfspaces, and their higher dimensional counterparts are represented. These topics would be covered in a more complete course on projective geometry or convexity.

**Explicit Representation:** If we think of a line as a linear function of the variable  $x$ , we can express any (nonvertical) line by the equation  $y = ax + b$ , where  $a$  is the slope and  $b$  is the  $y$ -intercept.

In dimension  $d$ , we can think of the  $d$ th coordinate as being special, and we will make the convention of referring to the  $d$ -th coordinate axis as pointing vertically upwards. We can express any “nonvertical”  $(d - 1)$ -dimensional hyperplane by the set of points  $(x_1, \dots, x_d)$ , where  $x_d = \sum_{i=1}^{d-1} a_i x_i + b$ , thus  $x_d$  is expressed “explicitly” as a linear function of the first  $d - 1$  coordinates.

The associated halfspaces arise replacing “=” with an inequality, e.g., the *upper halfplane* is the set  $(x, y)$  such that  $y \geq ax + b$ , and the *lower halfplane* is defined analogously.

**Implicit Representation:** The above representation has the shortcoming that it cannot represent vertical objects. A more general approach (which works for both hyperplanes and curved surfaces) is to express the object implicitly as the zero-set of some function of the coordinates. In the case of a line in the plane, we can represent the line as the set of points  $(x, y)$  that satisfy the linear function  $f(x, y) = 0$ , where  $f(x, y) = ax + by + c$ , for scalars  $a$ ,  $b$ , and  $c$ . The corresponding halfplanes are just the sets of points such that  $f(x, y) \geq 0$  and  $f(x, y) \leq 0$ .

This has the advantage that it can represent any line in the Euclidean plane, but the representation is not unique. For example, the line described by  $5x - 3y = 2$  is the same as the line described by  $10x - 6y = 4$ , or any scalar multiple thereof. We could apply some normalization to overcome this, for example, by requiring that  $a^2 + b^2 = 1$ . If this normalization is performed, then the line has the convenient geometric interpretation that it is orthogonal to the unit vector  $(a, b)$  and is at distance  $c$  from the origin.

**Parametric Representation:** We can generalize the above to arbitrary dimensions. For example, in  $\mathbb{R}^d$ , the equation  $\sum_{i=1}^d a_i x_i = b$  describes a  $(d - 1)$ -dimensional hyperplane. What if you want to represent a line, or more generally, a flat object any lower dimension? For  $1 \leq k \leq d - 1$ , we can represent any  $k$ -dimensional “flat” object (called a  $k$ -flat or a  $k$ -dimensional affine subspace) as the affine span of a set of  $(k + 1)$  points. For example, we can represent any line in 3-dimensional space by taking any two distinct points  $p$  and  $q$  on the line, and then considering all the affine combinations  $(1 - \alpha)p + \alpha q$ , for  $\alpha \in \mathbb{R}$ . More generally, given any  $k + 1$  points  $\{p_1, \dots, p_k\}$ , we can represent any point on their span as  $\sum_{i=1}^{k+1} \alpha_i p_i$ , where  $\sum_{i=1}^{k+1} \alpha_i = 1$ .

This is called the *parametric representation*, since each point on the object is identified through the value of the parameter  $\alpha$  (or generally the parameters  $\alpha_1, \dots, \alpha_{k+1}$ ).

**Divide-and-Conquer Algorithm:** Returning to the halfplane intersection problem, recall that we are given a set  $H = \{h_1, \dots, h_n\}$  of halplanes and wish to compute their intersection. Here is a simple divide-and-conquer algorithm.

- (1) If  $n = 1$ , then just return this halfplane as the answer.
- (2) Otherwise, partition  $H$  into subsets  $H_1$  and  $H_2$ , each of size roughly  $n/2$ .
- (3) Compute the intersections  $K_1 = \bigcap_{h \in H_1} h$  and  $K_2 = \bigcap_{h \in H_2} h$  recursively.
- (4) If either either  $K_1$  or  $K_2$  is empty, return the empty set. Otherwise, compute the intersection of the convex polygons  $K_1$  and  $K_2$  (by the procedure described below).

If we let  $I(n)$  denote the time needed to intersect two convex polygons, each with at most  $n$  vertices, we obtain the following recurrence for the overall running time:

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2T(n/2) + I(n) & \text{if } n > 1, \end{cases}$$

We will show below that  $I(n) \leq cn$ , for some constant  $c$ . It follows by standard results (consult the *Master Theorem* in CLRS) that  $T(n)$  is  $O(n \log n)$ .

**Intersecting Two Convex Polygons:** The only remaining task is the process of intersecting two convex polygons,  $K_1$  and  $K_2$  (see Fig. 2(a)). Note that these are somewhat special convex polygons because they may be empty or unbounded.

We can compute the intersection by a left-to-right plane sweep in  $O(n)$  time (see Fig. 2(b)). We begin by breaking the boundaries of the convex polygons into their upper and lower chains. (This can be done in  $O(n)$  time.) By convexity, the sweep line intersects the boundary of each convex polygon  $K_i$  in at most two points, one for the upper chain and one for the lower chain. Hence, the sweep-line status contains at most four points. This implies that updates to the sweep-line status can be performed in  $O(1)$  time. Also, we need keep track of a constant number of events at any time, namely the right endpoints of the current segments in the sweep-line status, and the intersections between consecutive pairs of segments. Thus, each step of the plane-sweep process can be performed in  $O(1)$  time.

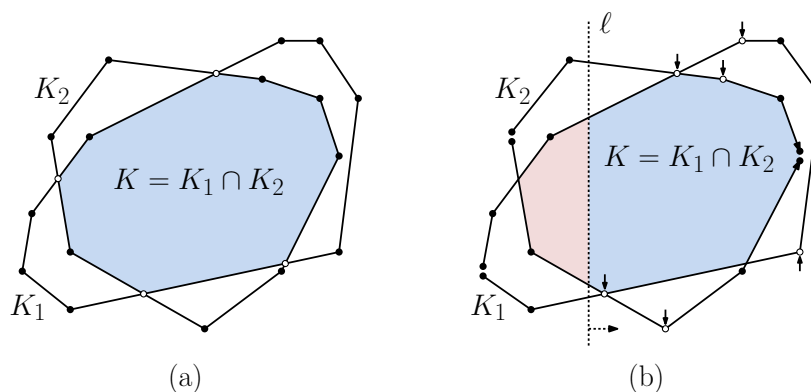


Fig. 2: Intersecting two convex polygons by plane sweep.

The total number of events is equal to the total number vertices, which is  $n$ , and the total number of intersection points. It is an easy exercise (which we leave to you) to prove that two convex polygons with a total of  $n$  sides can intersect at most  $O(n)$  times. Thus, the overall running time is  $O(n)$ .

**Upper and Lower Envelopes of Lines:** Let's next consider a variant of the halfplane intersection problem. Given any set of nonvertical lines  $L = \{\ell_1, \ell_2, \dots, \ell_n\}$  in the plane. Each line defines two natural halfplanes, and upper and lower halfplane. The intersection of all the lower halfplanes is called the *lower envelope* of  $L$  and the *upper envelope* is defined analogously (see Fig. 3). Let's assume that each line  $\ell_i$  is given explicitly as  $y = a_i x - b_i$ .

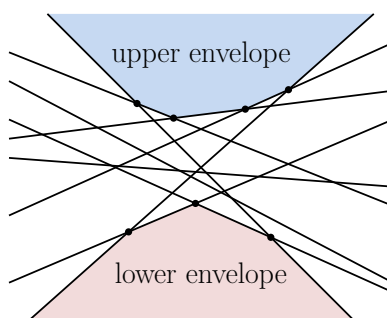


Fig. 3: Lower and upper envelopes.

The lower envelope problem is a restriction of the halfplane intersection problem, but it an interesting restriction. Notice that any halfplane intersection problem that does not involve any vertical lines can be rephrased as the intersection of two envelopes, a lower envelope defined by the lower halfplanes and an upper envelope defined by the upward halfplanes.

We will see that solving the lower envelope problem is very similar (in fact, essentially the same as) solving the upper convex hull problem. Indeed, they are so similar that exactly the same algorithm will solve both problems, without changing even a single character of code! All that changes is the way in which you interpret the inputs and the outputs.

**Motivating Duality:** Consider any line  $\ell : y = cx - d$  in the plane. Just as with point, it takes two real numbers ( $c$  and  $d$ ) to define any line in  $\mathbb{R}^2$ . Thus, we can identify the lines in  $\mathbb{R}^2$  with the set of points  $(c, d)$  in an alternative or “dual” space. For example, the line  $\ell : y = 2x + 1$  corresponds to the point  $(2, -1)$  in dual space, which we denote by  $\ell^*$ . Conversely, each point  $p = (a, b)$  is associated with a dual line,  $p^* : y = ax - b$ , which we denote by  $p^*$ .

This insight would not be of much use unless we could say something about how geometric relationships in one space relate to the other. The connection between the two involves incidences between points and line.

| Primal Relation                         | Dual Relation                               |
|---|---|
| Two (nonparallel) lines meet in a point | Two points join to form a line              |
| A point lies on a line                  | A line passes through a point               |
| A point lies above/below a line         | A line passes above/below a point           |
| Three points may be collinear           | Three lines may pass through the same point |

We’ll show that these relationships are preserved by duality. For example, consider the two lines  $\ell_1 : y = 2x + 1$  and the line  $\ell_2 : y = -\frac{x}{2} + 6$  (see Fig. 4(a)). These two lines intersect at the point  $p = (2, 5)$ . The duals of these two lines are  $\ell_1^* = (2, -1)$  and  $\ell_2^* = (-\frac{1}{2}, -6)$ . The line in the  $(a, b)$  dual plane passing through these two points is easily verified to be  $b = 2a - 5$ . Observe that this is exactly the dual of the point  $p$  (see Fig. 4(b)). (As an exercise, prove this for two general lines.)

**Dual Transformation:** Let us explore this dual transformation more formally. Duality (or more specifically *point-line duality*) is a transformation that maps points in the plane to lines and

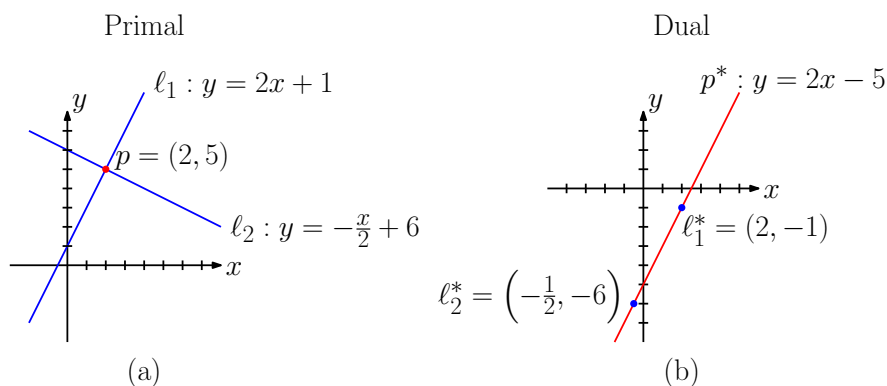


Fig. 4: The primal and dual planes.

lines to point. Given point  $p = (a, b)$  and line  $\ell : y = cx - d$ , we define<sup>1</sup>

$$p^* : y = ax - b \quad \text{and} \quad \ell^* = (c, d).$$

**Properties of Point-Line Duality:** Duality has a number of interesting properties, each of which is easy to verify by substituting the definition and a little algebra.

**Self Inverse:**  $p^{**} = p, \ell^{**} = \ell$  (This is trivial.)

**Incidence Preserving:** Point  $p$  is on line  $\ell$  if and only if point  $\ell^*$  is on line  $p^*$ . (Proof: The first assertion is equivalent to  $b = ac - d$  and the second is equivalent to  $d = ca - b$ . Clearly, one is satisfied if and only if the other is.)

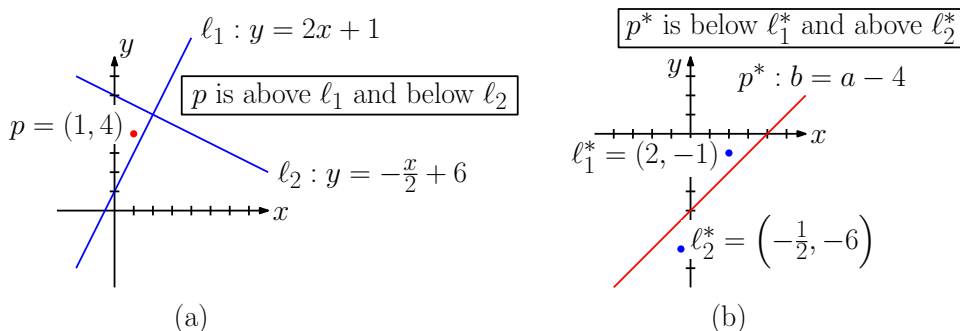


Fig. 5: Order reversal.

**Order Reversal:** Point  $p$  is above/below line  $\ell$  if and only if line  $p^*$  is below/above point  $\ell^*$ , respectively (see Fig. 5). (Proof: The first assertion is equivalent to  $b > ac - d$  and the second to  $d > ca - b$ . Clearly, one is satisfied if and only if the other is.)

<sup>1</sup>Duality can be generalized to higher dimensions as well. In  $\mathbb{R}^d$ , let us identify the  $y$  axis with the  $d$ -th coordinate vector, so that an arbitrary point can be written as  $p = (x_1, \dots, x_{d-1}, y)$  and a  $(d-1)$ -dimensional hyperplane can be written as  $h : y = \sum_{i=1}^{d-1} a_i x_i - b$ . The dual of this hyperplane is  $h^* = (a_1, \dots, a_{d-1}, b)$  and the dual of the point  $p$  is  $p^* : b = \sum_{i=1}^{d-1} x_i a_i - y$ . All the properties defined for point-line relationships generalize naturally to point-hyperplane relationships, where notions of above and below are based on the assumption that the  $y$  (or  $b$ ) axis is “vertical.”

**Intersection Preserving:** Lines  $\ell_1$  and  $\ell_2$  intersect at point  $p$  if and only if the dual line  $p^*$  passes through points  $\ell_1^*$  and  $\ell_2^*$ . (Follows directly from incidence preservation.)

**Collinearity/Coincidence:** Three points are collinear in the primal plane if and only if their dual lines intersect in a common point. (Follows directly from incidence preservation.)

**Convex Hulls and Envelopes:** Let us return now to the question of the relationship between convex hulls and the lower/upper envelopes of a collection of lines in the plane. The following lemma demonstrates the, under the duality transformation, the convex hull problem is dually equivalent to the problem of computing lower and upper envelopes.

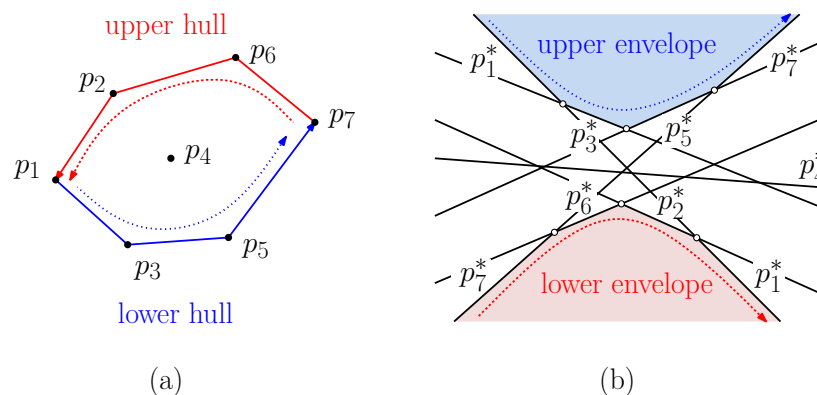


Fig. 6: Equivalence of hulls and envelopes.

**Lemma:** Let  $P$  be a set of points in the plane. The counterclockwise order of the points along the upper (lower) convex hull of  $P$  (see Fig. 6(a)), is equal to the left-to-right order of the sequence of lines on the lower (upper) envelope of the dual  $P^*$  (see Fig. 6(b)).

**Proof:** We will prove the result just for the upper hull and lower envelope, since the other case is symmetrical. For simplicity, let us assume that no three points are collinear.

Consider a pair of points  $p_i$  and  $p_j$  that are consecutive vertices on the upper convex hull. This is equivalent to saying that all the other points of  $P$  lie beneath the line  $\ell_{ij}$  that passes through both of these points.

Consider the dual lines  $p_i^*$  and  $p_j^*$ . By the incidence preserving property, the dual point  $\ell_{ij}^*$  is the intersection point of these two lines. (By general position, we may assume that the two points have different  $x$ -coordinates, and hence the lines have different slopes. Therefore, they are not parallel, and the intersection point exists.)

By the order reversing property, all the dual lines of  $P^*$  pass above point  $\ell_{ij}^*$ . This is equivalent to saying the  $\ell_{ij}^*$  lies on the lower envelope of  $P^*$ .

To see how the order of points along the hulls are represented along the lower envelope, observe that as we move counterclockwise along the upper hull (from right to left), the slopes of the edges increase monotonically. Since the slope of a line in the primal plane is the  $a$ -coordinate of the dual point, it follows that as we move counterclockwise along the upper hull, we visit the lower envelope from left to right.

One rather cryptic feature of this proof is that, although the upper and lower hulls appear to be connected, the upper and lower envelopes of a set of lines appears to consist of two

disconnected sets. To make sense of this, we should interpret the primal and dual planes from the perspective of *projective geometry*, and think of the rightmost line of the lower envelope as “wrapping around” to the leftmost line of the upper envelope, and vice versa. The places where the two envelopes wraps around correspond to the vertical lines (having infinite slope) passing through the left and right endpoints of the hull. (As an exercise, can you see which is which?)

**Primal/Dual Equivalencies:** There are a number of computational problems that are defined in terms of affine properties of point and line sets. These can be expressed either in primal or in dual form. In many instances, it is easier to visualize the solution in the dual form. We will discuss many of these later in the semester. For each of the following, can you determine what the dual equivalent is?

- Given a set of points  $P$ , find the narrowest slab (that is, a pair of parallel lines) that contains  $P$ . Define the width of the slab to be the vertical distance between its bounding lines.
- Given a convex polygon  $K$ , find the longest vertical line segment with one endpoint on  $K$ 's upper hull and one on its lower hull.
- Given a set of points  $P$ , find the triangle of smallest area determined by any three points of  $P$ .