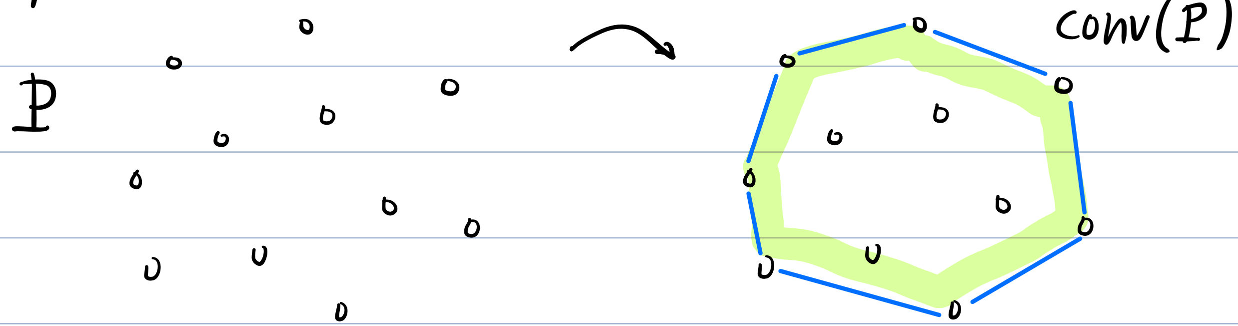
CMSC 754 - Computational Geometry
Lecture 2: Convex Hulls in the Plane

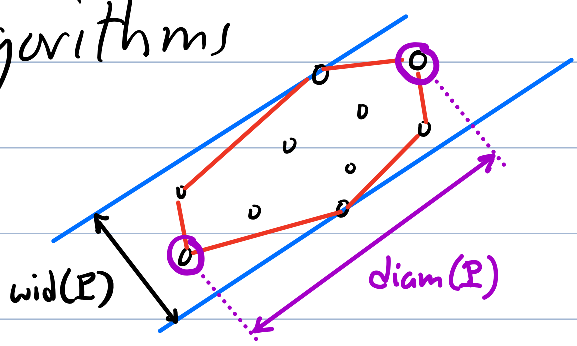
Convex Hull: (Intuitive definition)

Given a point set P in $\mathbb{R}^2$, imagine snapping a rubber band around the points



Uses:

- shape approximation (intersection test)
- first step in other algorithms
  - diameter
  - width



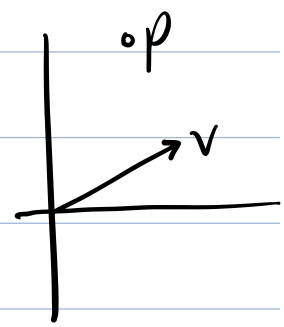
Basic Definitions:

$\mathbb{R}^d$ - Real d-dim space $p = (p_1, \dots, p_d)$ $p_i \in \mathbb{R}$

- Refer to as

points (p, q) - location

or vectors (u, v, w) - displacement



$\mathbb{R}$ - scalars $\alpha, \beta, \gamma, \dots$

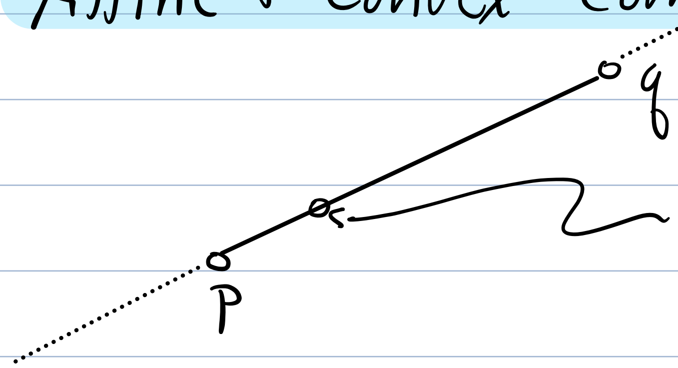
usual ops from linear algebra:

$u+v$, $u-v$ - vector addition

$\alpha \cdot u$ - scalar multiplication

$u \cdot v$ - dot product $= \sum_{i=1}^{d} u_i v_i$

Affine + Convex Combinations:



for $\alpha \in \mathbb{R}$

$(1-\alpha)p + \alpha q$

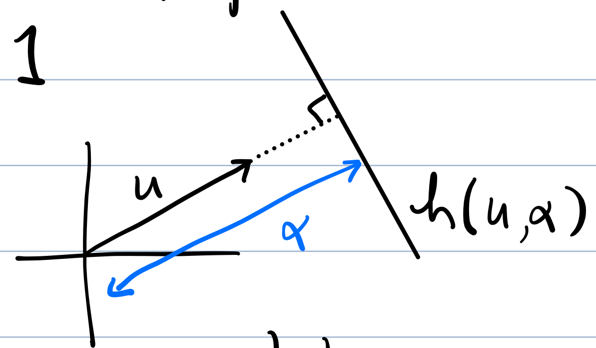
Generally given $p_1 \dots p_k$:

Affine combination: $\sum_{i=1}^{k} \alpha_i p_i$ $\sum \alpha_i = 1$

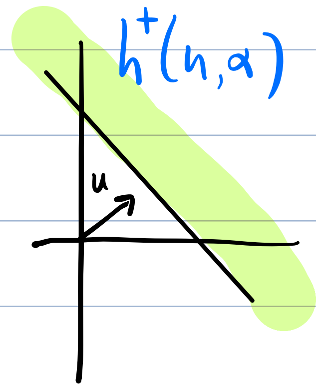
Convex combination: … and $0 \le \alpha_i \le 1$

Lines, Hyperplanes, Halfspaces:

Given nonzero vector u + scalar α,
$h(u,\alpha) = \{ p \in \mathbb{R}^d \mid p \cdot u = \alpha \}$ is hyperplane
If $\|u\| = 1$



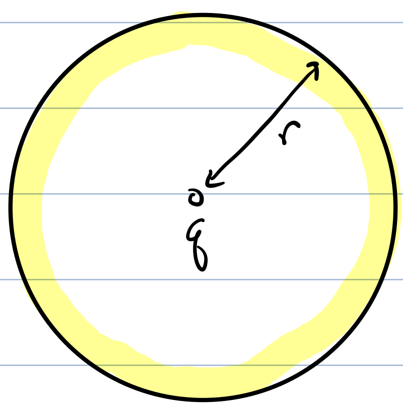
$h^+(u,\alpha) = \{ p \in \mathbb{R}^d \mid p \cdot u \geq \alpha \}$



Euclidean Ball:

$$\text{dist}(p,q) = \|p-q\| = \left(\sum_{i=1}^{d} (p_i - q_i)^2 \right)^{1/2}$$

$B(q,r) = \{ p \in \mathbb{R}^d \mid \|p-q\| \leq r \}$
(Euclidean) ball of radius r centered at q.



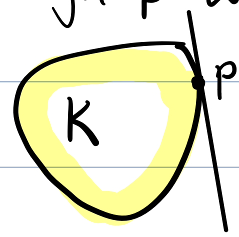
Convexity:

A set $K \subseteq \mathbb{R}^d$ is **convex** if $\forall\, p,q \in K$ the line segment $\overline{pq}$ (equiv. any conv. combination of $p + q$) lies within K

Support Hyperplane:

Boundary of K

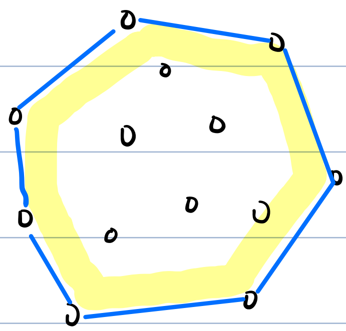
Given convex K and any point $p \in \partial K$, $\exists$ hyperplane passing through p with K lying all on one side.



Convex Hull:

Given a set P of points in $\mathbb{R}^d$, the convex hull, $\text{conv}(P)$, is the smallest convex set containing P.

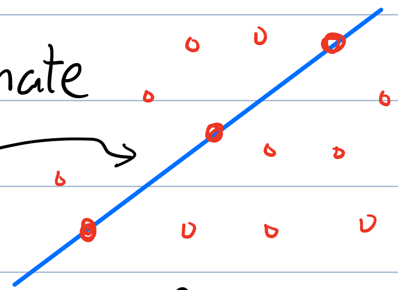
- The set of all convex combs in P
- The intersection of all halfspaces containing P



General Position:

Geometric algorithms are complicated by rare (?) degenerate cases:

- points having same coordinate
- ≥ 3 collinear points
- ≥ 4 cocircular points



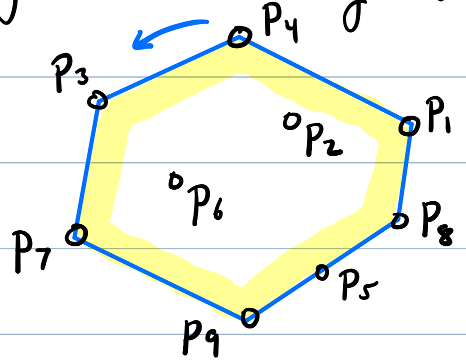
To simplify algorithm presentation we often assume these do not arise in the input.

Called **general-position assumption**

(Planar) Convex Hull Problem:

Given a set of n pts $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$ ($p_i = (x_i, y_i)$) compute $\mathrm{conv}(P)$.

Output: Cyclic ordering of vertices on the hull



possible output: (indices)

$\langle 4, 3, 7, 9, 8, 1 \rangle$

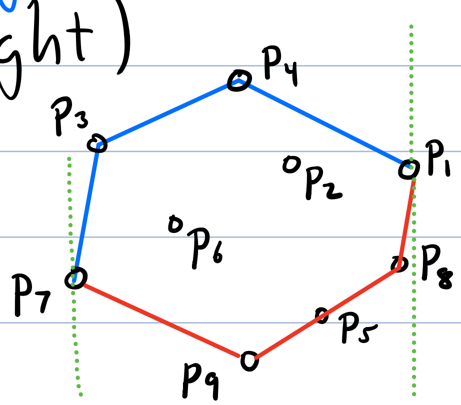
Note: p_5 not output

(Can assume this away by "general position")

Alternative output: (left to right)

Upper-hull + **Lower-hull**

$\langle 7, 3, 4, 1 \rangle + \langle 7, 9, 8, 1 \rangle$

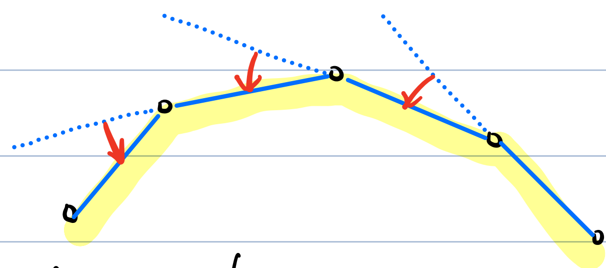


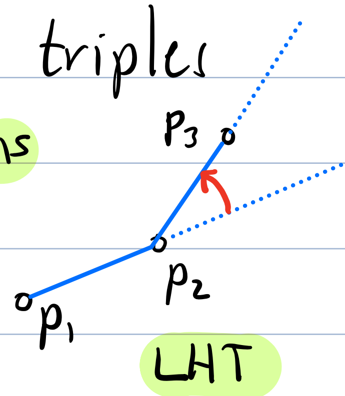
Graham's Scan: $O(n \log n)$ solution

- Compute upper + lower hulls separately
- Upper-hull:
  - Sort pts by x-coords
  - Add each to upper hull
  - Remove pts no longer on hull ← How?
- Lower-hull: (symmetrical)

Observations:

- The rightmost pt always on hull
- Reading right to left, consecutive triples on the hull form left-hand turns

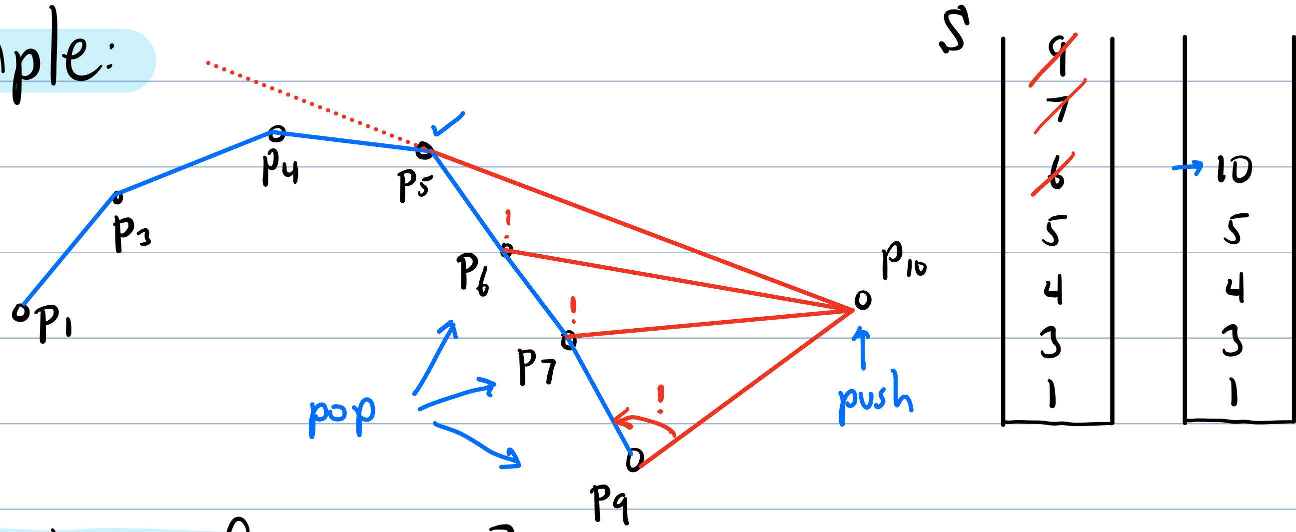




Incremental Approach:

- Store vertices (indices) of upper hull on stack
- For each new point p_i (left to right)
  - While $\langle p_i, S[top], S[top-1] \rangle$ do not form LHT — pop S
  - Push p_i

Example:



How to test for LHT?

Orientation test

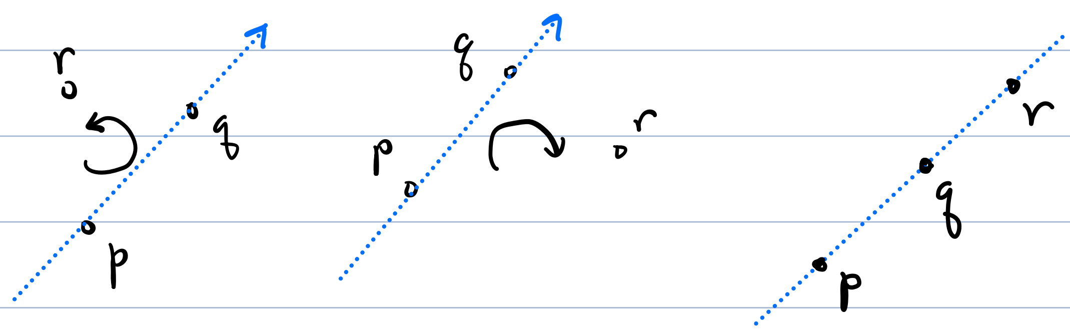
Given a sequence $\langle p, q, r \rangle$ of 3 pts in $\mathbb{R}^2$

$$\text{orient}(p,q,r) = \text{sign}\left(\det\begin{pmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{pmatrix}\right)$$

is: +1 if they are oriented CCW (LHT)

-1 " " " " CW (RHT)

0 if they are collinear (or duplicates)



+1 -1 0

Graham's Scan: (Upper Hull only)

- Sort pts by increasing x-coords $\langle p_1 \ldots p_n \rangle$
- Push p_1 + p_2 onto S
- for $i \leftarrow 3$ to n
  - while ($|S| \geq 2$ and orient($p_i, S[t], S[t-1]$) ≤ 0) pop S
  - push p_i

t = "top"

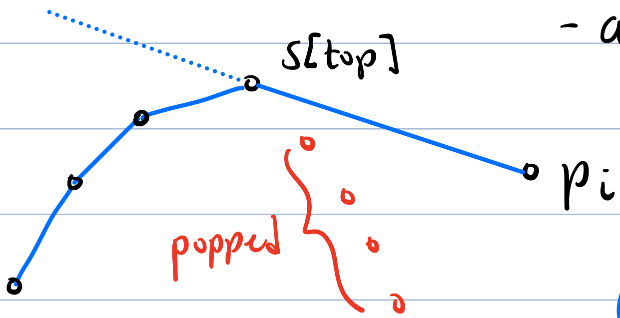
Correctness: (Sketch)

Lemma: After processing p_i, S contains upper hull of $\langle p_1 \ldots p_i \rangle$

Proof: By induction on i.

- p_i must be last vertex of hull
- all the popped pts are not on upper hull
  - all remaining pts are on upper hull (up to p_i)

(omit the details)



Running time:

- $O(n \log n)$ to sort
- for $3 \le i \le n$, let d_i = num. of pops when inserting p_i
- Time for scan is ~

$$\sum_{i=3}^{n} (\underbrace{d_i}_{\text{for pops}} + \underbrace{1}_{\text{for push of } p_i}) \le n + \sum_{i=3}^{n} d_i$$

- Note that $\sum_i d_i \le n$ → Why?

- Total time: $O(n \log n + 2n) = O(n \log n)$

Also see lecture notes for a hull algorithm based on divide + conquer