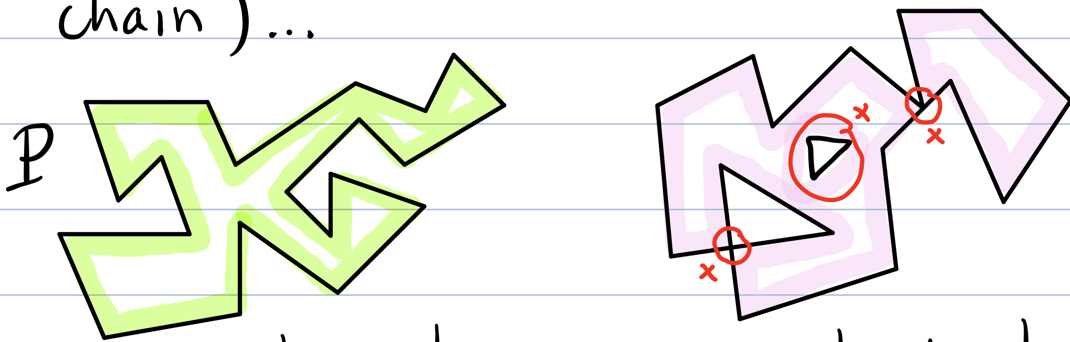
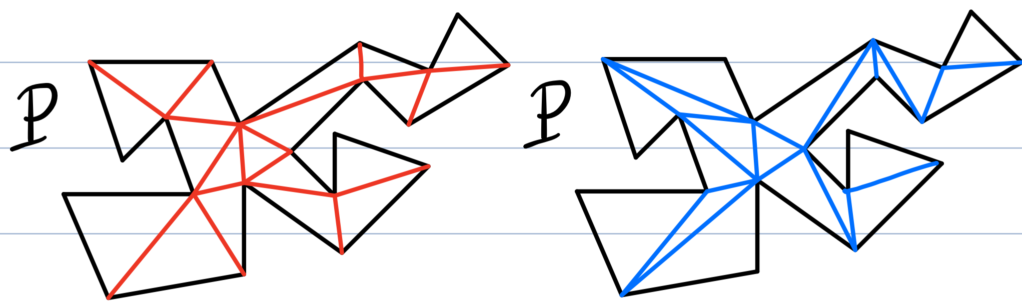
CMSC 754 - Computational Geometry
Lecture 5: Polygon Triangulation

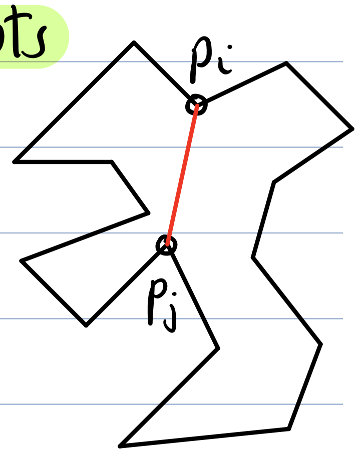
Polygon Triangulation: Given a simple polygon P (that is, a simple, closed polygonal chain)...



subdivide the interior of P into triangles (vertices drawn from P's vertices)



Notes: - P given as a cyclic seq. of pts
- Vertices p_i + p_j are visible if open segment $\overline{p_i p_j} \subseteq \text{int}(P)$
- If p_i + p_j visible, segment $\overline{p_i p_j}$ called a diagonal



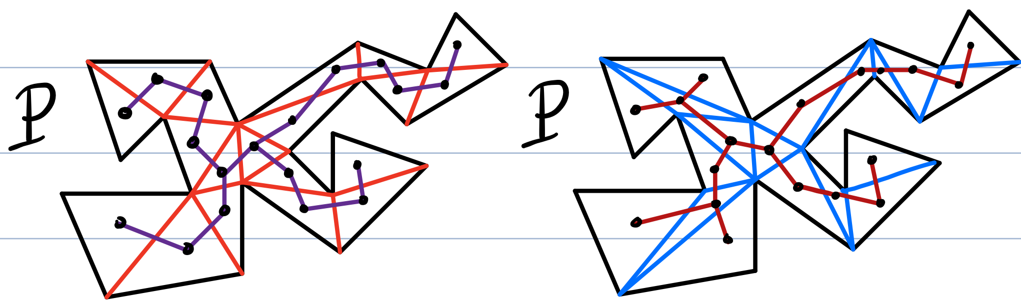
Lemma: Given any n-vertex simple polygon ($n \geq 3$)

- A triangulation exists
- Any triangulation has $n-3$ diagonals
- Any triangulation has $n-2$ triangles

Dual Graph: A triangulation defines a graph:

Vertices ← triangles

Edges ← adjacent (share common edge)



The dual graph of a polygon triangulation is connected + acyclic ⇒ tree

History of Polygon Triangulation:

$O(n^2)$ - Easy (find a diagonal + recurse)

$O(n \log n)$ - We'll present this

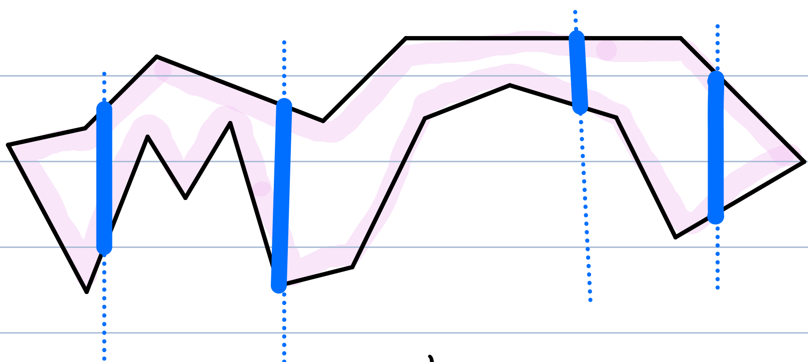
$O(n)$ - Chazelle 1991 (very complicated!)

Two steps:

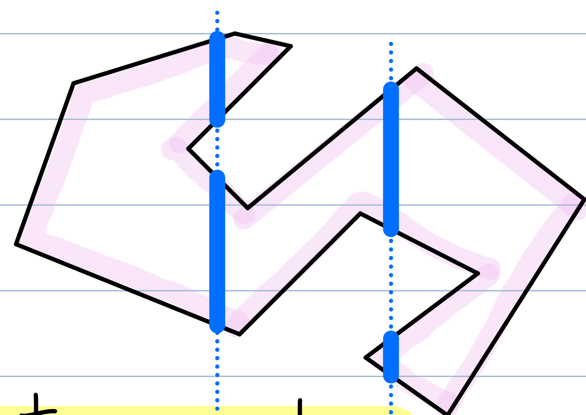
1. Decompose the polygon into (simpler) polygons – monotone polygons – $O(n \log n)$
2. Triangulate each monotone polygon – $O(n)$

Output: Graph structure, called a doubly-connected edge list (DCEL)

Def: A polygon is x-monotone if any vertical intersects the polygon in a single segment (if at all)

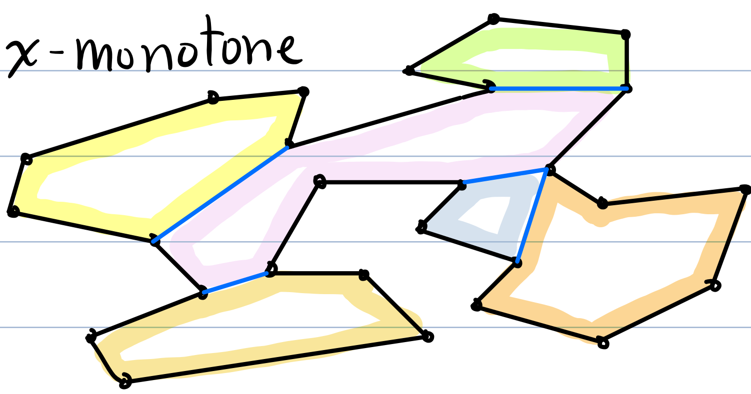


x-monotone



not x-monotone

Monotone Decomposition – Add (non-intersecting) diagonals so that connected components are all x-monotone

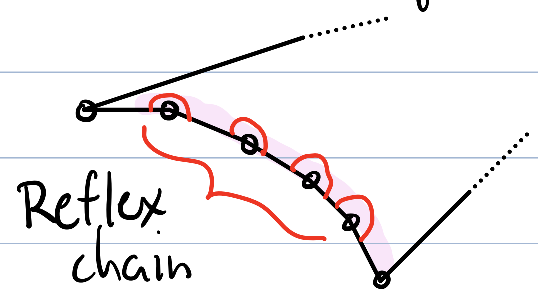


Triangulating a Monotone Polygon:

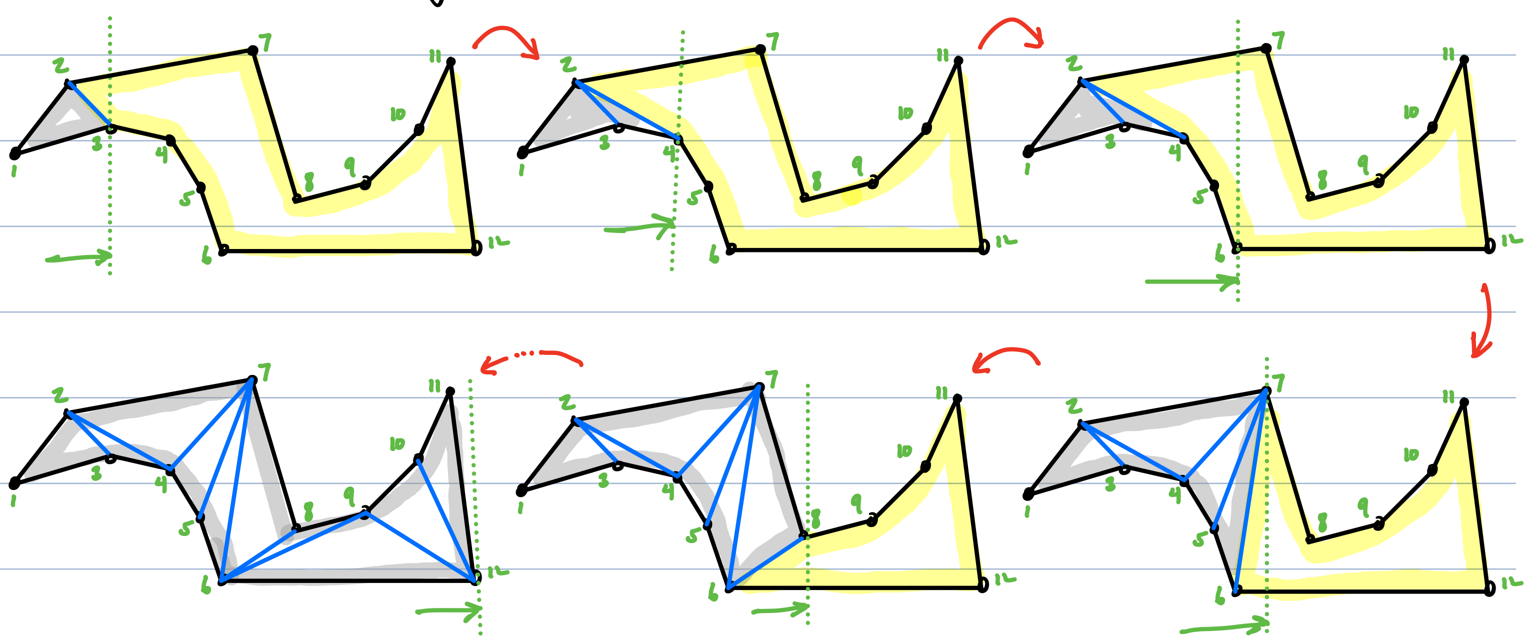
General position: No duplicate x-coords (no vertical edges)

Reflex Vertex: Internal angle $\geq \pi$

Reflex Chain: Sequence of reflex vertices



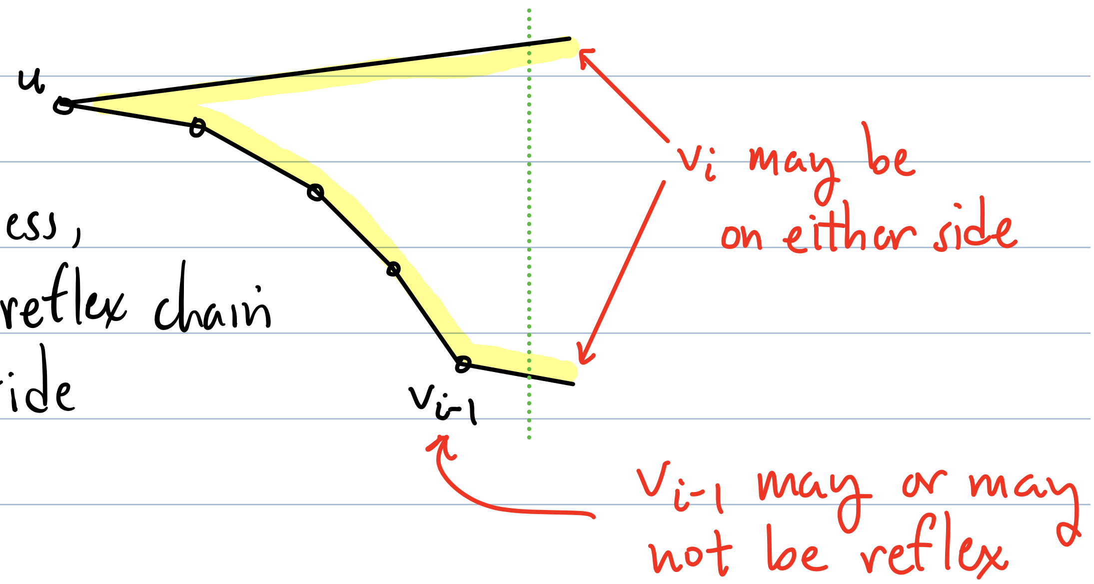
General approach: Sweep from left to right + triangulate as much as we can behind us.



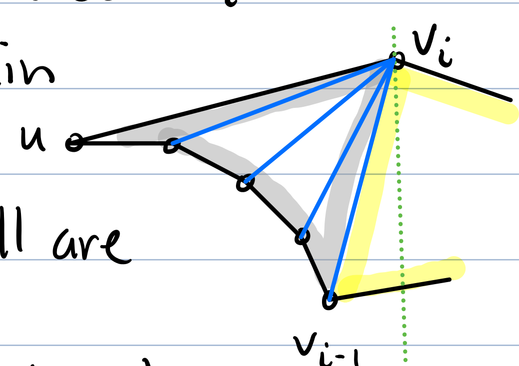
What's the loop invariant?

Lemma: For $i \geq 2$, let v_i be the next vertex to process. The untriangulated region to left of v_i consists of two x-monotone chains starting from a common vertex u. One chain is a single edge, and the other is a reflex chain (of one or more edges).

For concreteness, let's assume reflex chain is on lower side



Case 1: (v_i lies on upper chain)
- add diagonals between v_i and all vertices of the chain



[By monotonicity, all are visible to v_i]

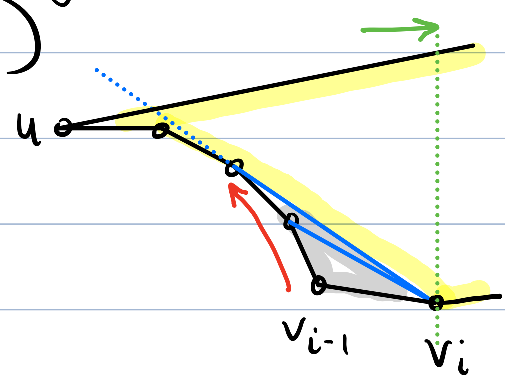
Now $u = v_{i-1}$. Reflex chain has just one edge.

Case 2: (v_i lies on lower chain)

2a: (v_{i-1} is non-reflex)

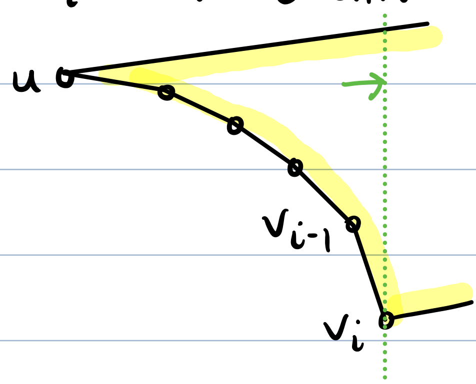
- connect v_i to all visible vertices on chain until hitting point of tangency. (Similar to Graham's scan)

[May go all the way back to u]



2b: (v_{i-1} is reflex)

- Add v_i to the chain



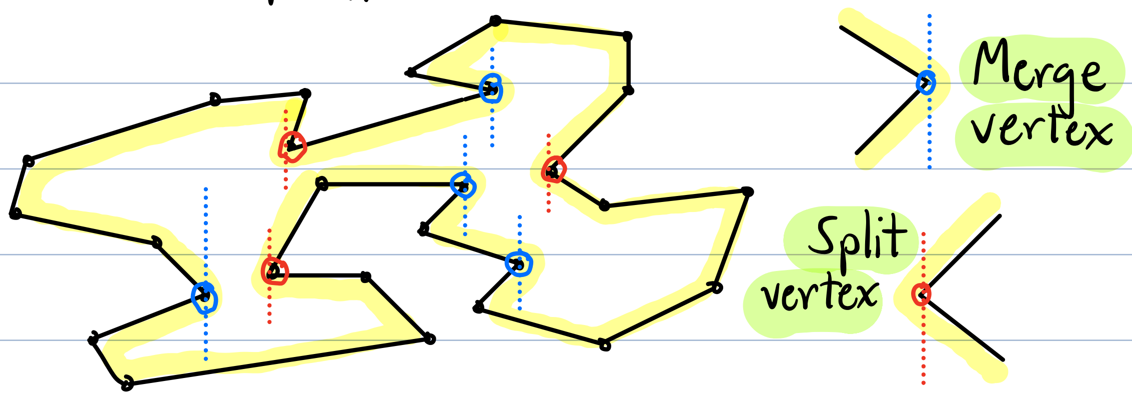
Correctness: Invariant holds after each iteration

Running time: $O(n)$ [As in Graham, once a vertex is removed from the chain, it never reappears)

Monotone Subdivision:

Recall: Add diagonals to create x-monotone

Where? Scan reflex vertex: Reflex vertex where both edges on same side of vertical line.



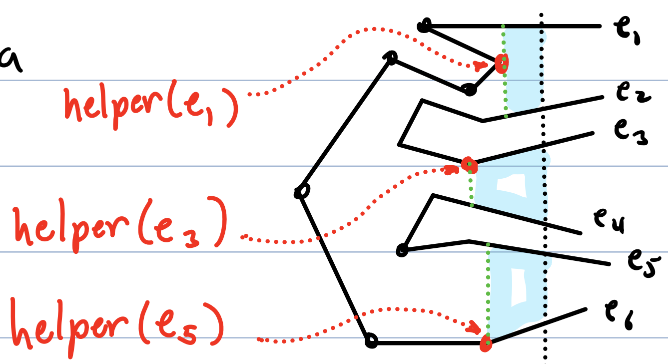
Add a diagonal to right side of each merge
" " " " left " " " split

Plane-sweep Approach:

Need auxilliary info to help with diagonals

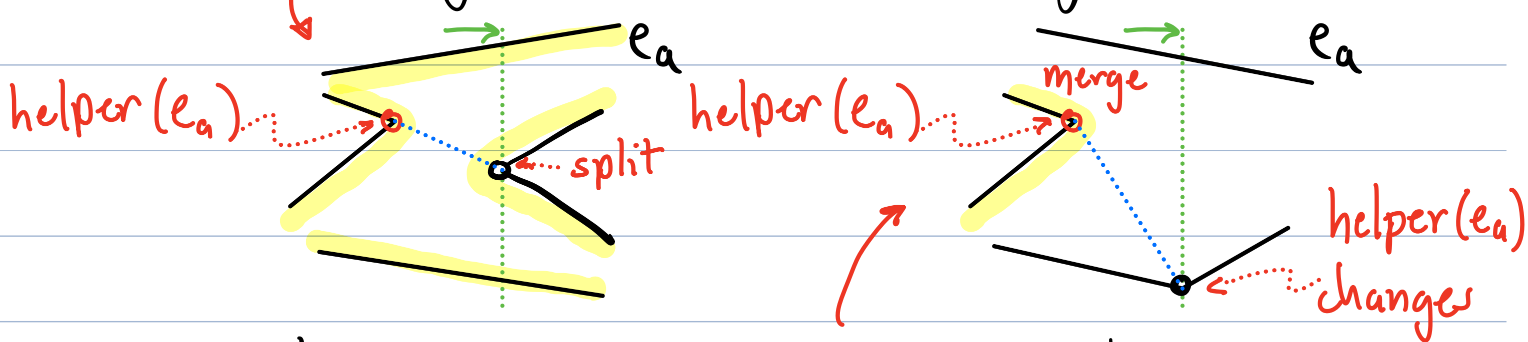
For each edge e_a of sweep line with int($\mathcal{P}$) below:

helper(e_a) = rightmost vertically visible vertex on or below e_a to left of sweep line



Why is the helper helpful?

- When we see a split vertex, we add diagonal to helper of edge above



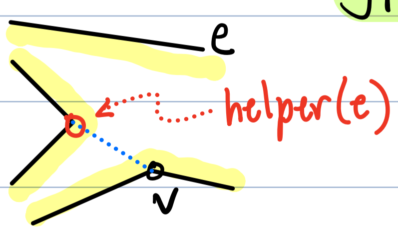
- When we see a merge vertex, it is the helper of edge above + we connect it to next vertex where helper(e_a) changes

Events: Polygon vertices (sorted by x)

Sweep-line status: Edges intersecting the sweep line (ordered dictionary)

Event processing: There are many cases!

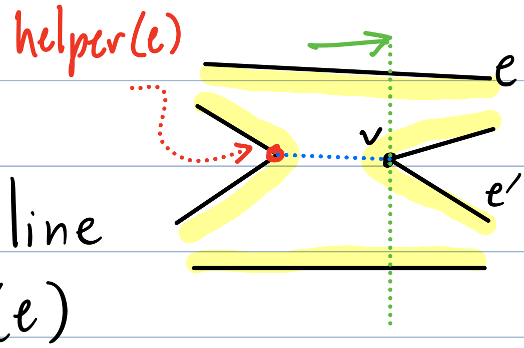
Utility:



fix-up(v, e):

if (helper(e) is a merge vertex)
    add diagonal v to helper(e)

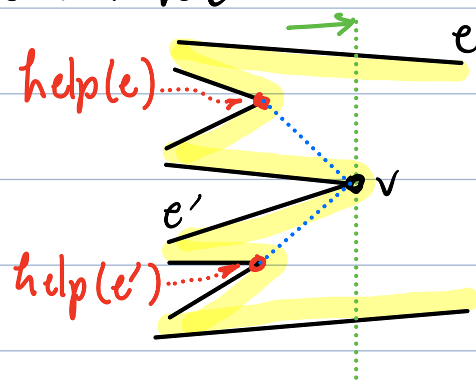
Split Vertex (v):



- e ← edge above v in sweep line
- add diagonal v to helper(e)
- insert edges incident to v into sweep line
- letting e′ be lower, set helper(e′) ← v

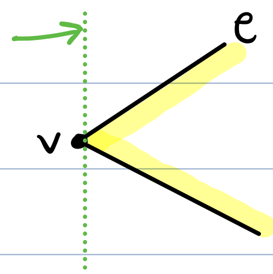
Merge Vertex (v):

- Consider two edges incident to v + let e′ be lower one



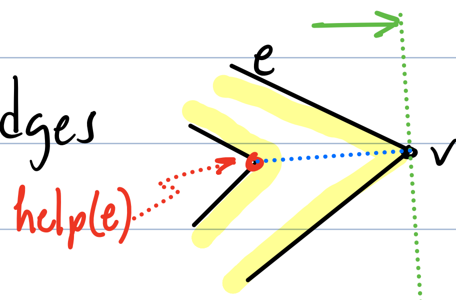
- Delete both from sweep line
- Let e be edge above v
- fix-up(v,e) + fix-up(v,e′)

Start vertex (v):



- Insert v's incident edges into sweep line
- Letting e be upper edge, helper(e) ← v

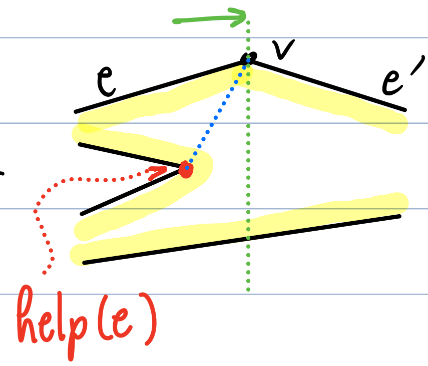
End vertex (v):



- Consider the two incident edges + let e be upper edge
- Delete both from sweep line
- fix-up(v,e)

Upper-chain vertex (v):

- Let e be edge to left, e′ to right
- fix-up(v, e)
- Replace e with e′ in sweep line
- helper(e′) ← v



Lower-chain vertex (v):

- Let e be edge above
- fix-up(v, e)
- Let e′ be edge to left, e″ to right
- Replace e′ with e″ in sweep line