

# CMSC 754 - Computational Geometry

## Lecture 6: Halfplane Intersection + Duality

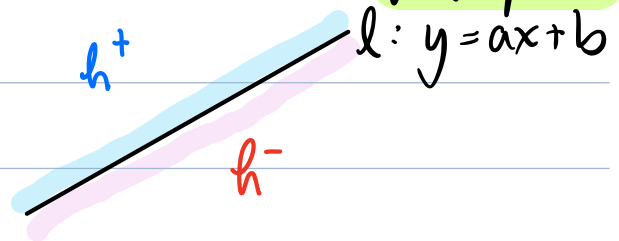
### Halfplane Intersection:

Recall, each line in plane defines two halfspaces

$$l: y = ax + b$$

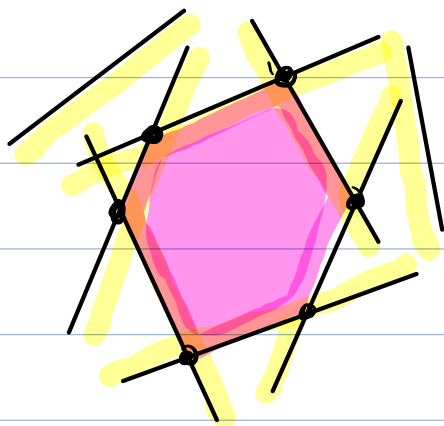
$$h^+: y \geq ax + b$$

$$h^-: y \leq ax + b$$

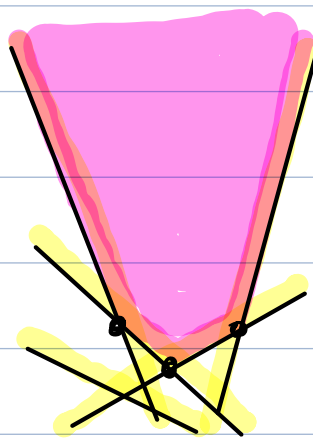


A halfspace is an (unbounded) convex set

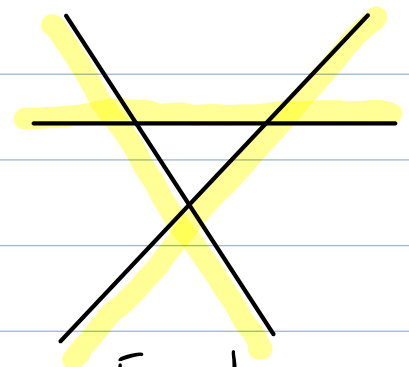
Given a set of halfspaces:  $H = \{h_1, \dots, h_n\}$   
their intersection  $\bigcap h_i$  is a (possibly unbounded / possibly "empty") convex polygon



Bounded



Unbounded



Empty

## Representing lines (and more):

$\mathbb{R}^2$  (Line)

$\mathbb{R}^d$  (Hyperplane)

Explicit:  
 $y = f(x)$

$$y = ax + b$$

$$x_d = \sum_{i=1}^{d-1} a_i x_i + b$$

Implicit:

$$f(x, y) = 0$$

$$f(x, y) = ax + by + c$$

$$f(x_1, \dots, x_d) = \sum_{i=1}^d a_i x_i + b$$

Parametric:

$$\begin{aligned} (x(t), y(t)) \\ t \in \mathbb{R} \end{aligned} \quad \begin{aligned} x(t) &= at + b \\ y(t) &= ct + d \end{aligned}$$

$$\begin{aligned} (x_1(t), \dots, x_d(t)) \\ x_i(t) &= a_i t + b_i \end{aligned}$$

↪ line in  $\mathbb{R}^d$

## Halfplane Intersection:

Given a set  $H = \{h_1, \dots, h_n\}$  of halfspaces, output  $\bigcap_{i=1}^n h_i$  (e.g. as vertices of conv. polygon)

Divide and Conquer Algorithm:  $O(n \log n)$

Intersect( $H$ ) {

- if ( $|H| = 1$ ) return  $h_1$  [single halfspace]

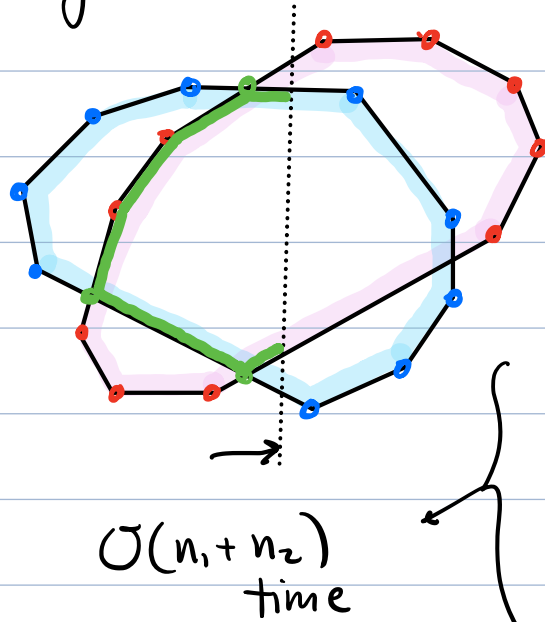
- else

partition  $H \begin{cases} \rightarrow H_1 \\ \rightarrow H_2 \end{cases} \quad |H_i| \leq \frac{n}{2}$

$I_1 \leftarrow \text{Intersect}(H_1) \quad I_2 \leftarrow \text{Intersect}(H_2)$

return merge( $I_1, I_2$ ) ↪ How?

# How to merge? Plane sweep



- At most 4 segments hit sweep line
- $\leq n_1 + n_2$  end pt events  
 $n_i = |H_i|$
- $\leq 2(n_1 + n_2)$  intersection events
- Boundaries are already sorted

## Overall Running Time:

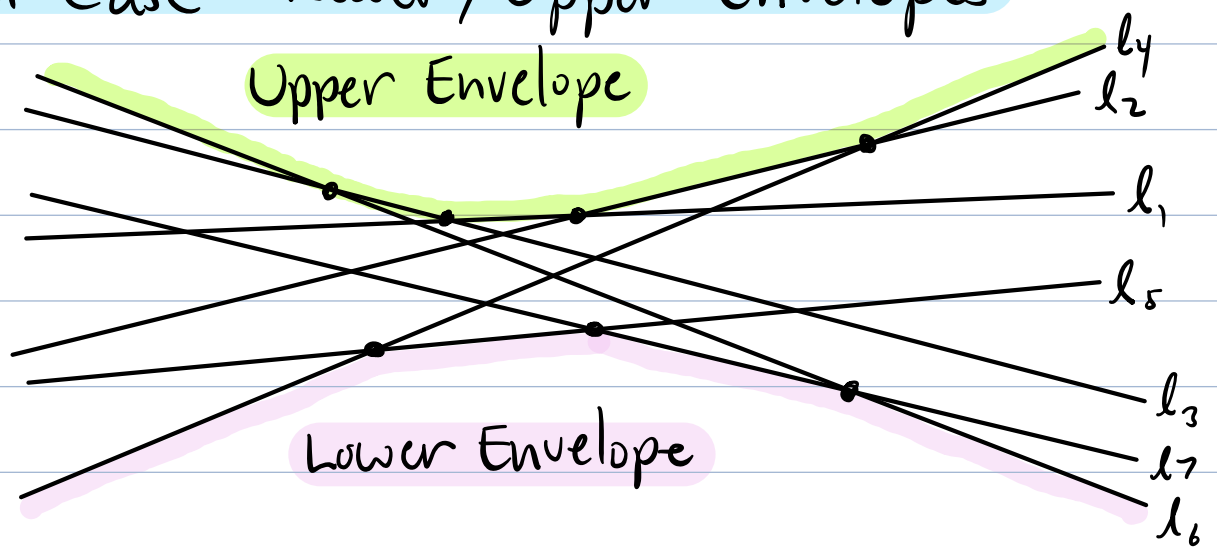
$$T(n) = 2T(n/2) + n$$

2 recursive calls on  $n/2$  halfspaces

merge in linear time

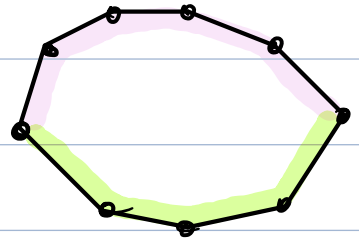
$$= O(n \log n) \quad [\text{see, eg., CLRS}]$$

## Special Case: Lower/Upper Envelopes



# Envelopes of lines $\sim$ Hull of points

Related?



## Point-Line Duality

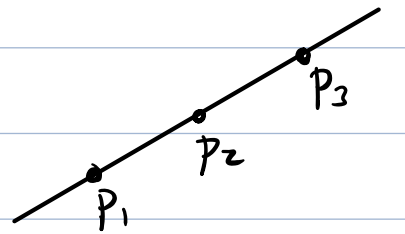
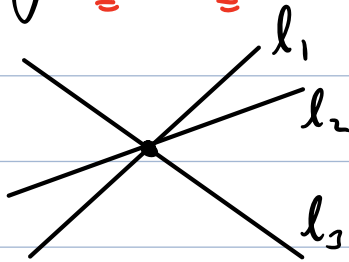
Lines in  $\mathbb{R}^2$  are a lot like points:

2 degrees of freedom

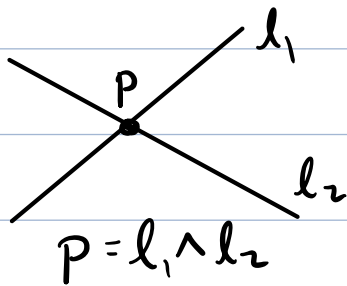
$$y = ax + b$$

$$p = (a, b)$$

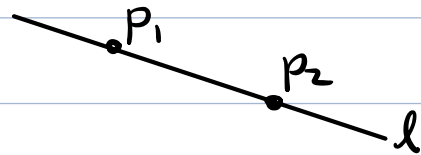
degeneracy:



incidence:



Two lines meet at a point



$$l = p_1 \vee p_2$$

Two points join to form a line

## Dual Operator:

Given point  $p = (a, b)$

$a, b \in \mathbb{R}$

line  $l: y = cx - d$

$c, d \in \mathbb{R}$

Dual  $p^*$  is the line  $y = a \cdot x - b$

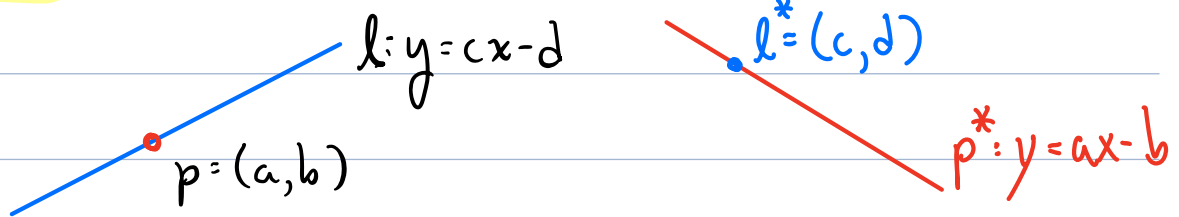
$l^*$  is the point  $(c, d)$

## Observations:

**Self-inverse:**  $p^{**} = p$   $l^{**} = l$

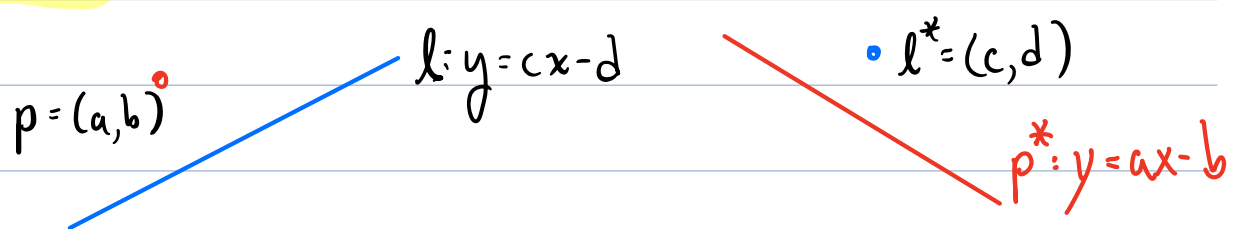
**Incidence:**  $p$  lies on  $l$  iff  $l^*$  lies on  $p^*$

**Proof:**  $b = c \cdot a - d \iff d = a \cdot c - b$



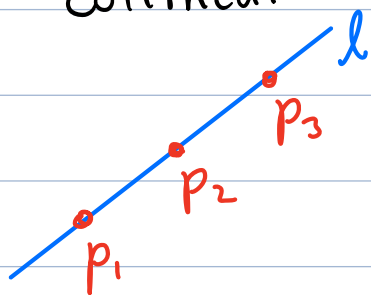
**Order reversing:**  $p$  lies above/below  $l$  iff  $p^*$  passes below/above  $l^*$

**Proof:**  $b > c \cdot a - d \iff d > a \cdot c - b$

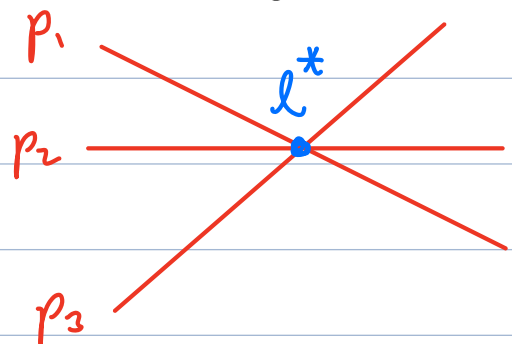


## Degeneracy:

$p_1, p_2, p_3$  are collinear



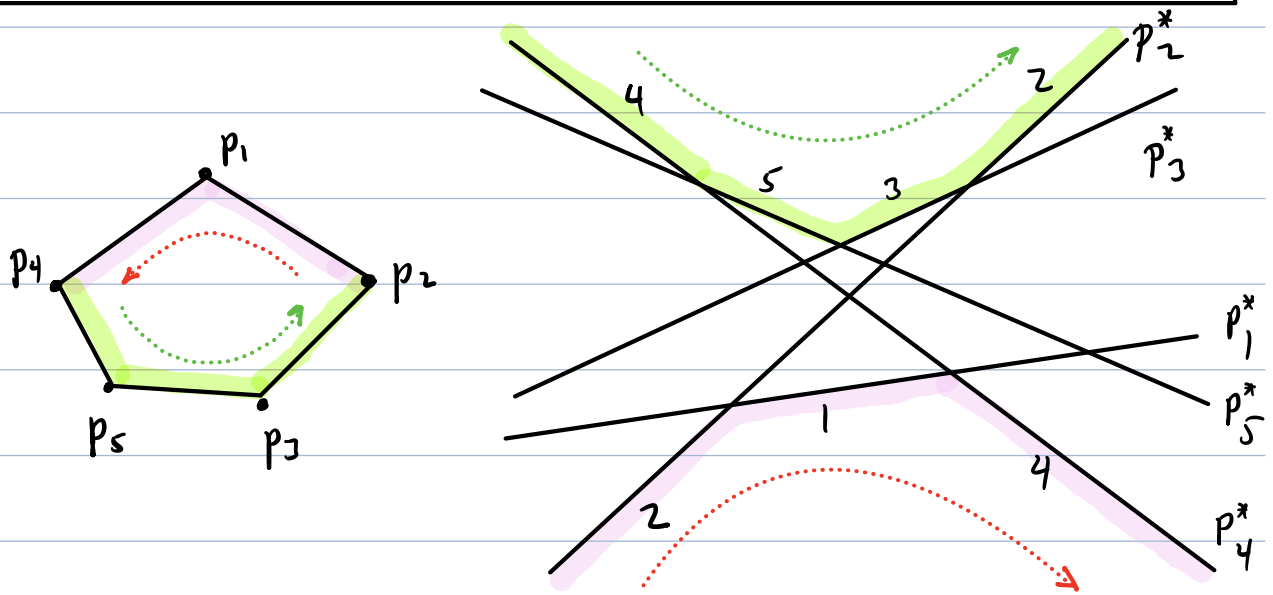
iff  $p_1^*, p_2^*, p_3^*$  are coincident



# Hulls and Envelopes:

Lemma:

Given a set  $P = \{p_1, \dots, p_n\}$  in  $\mathbb{R}^2$ , the CCW order of points on  $P$ 's upper/lower hull is same as left-right order of segments in  $P^*$ 's lower/upper envelope



Proof: (Sketch)

Consider edge  $p_i p_j$  on upper hull of  $\text{conv}(P)$

Let  $l$  be line  $\overleftrightarrow{p_i p_j}$  - All pts of  $P$  lie on or below  $l$

$\Leftrightarrow$  (order reversal) - All lines of  $P^*$  pass on or above point  $l^*$

$\Leftrightarrow l^*$  is vertex of lower envelope