CMSC 754 - Computational Geometry
Lecture 10: Voronoi Diagrams

Metric Spaces: Distances modeled as metric space (X, f): $f: X \times X \to \mathbb{R}^{\geq 0}$, s.t. for all $p, q, r \in X$:

- **Symmetry:** $f(p,q) = f(q,p)$
- **Positivity:** $f(p,q) \geq 0$ and $f(p,q) = 0$ iff $p = q$
- **Triangle Inequality:** $f(p,q) \leq f(p,r) + f(r,q)$

Euclidean Distance: for $p, q \in \mathbb{R}^d$:

$$\|p - q\| = \left[\sum_i (p_i - q_i)^2 \right]^{1/2}$$

Voronoi Diagram:

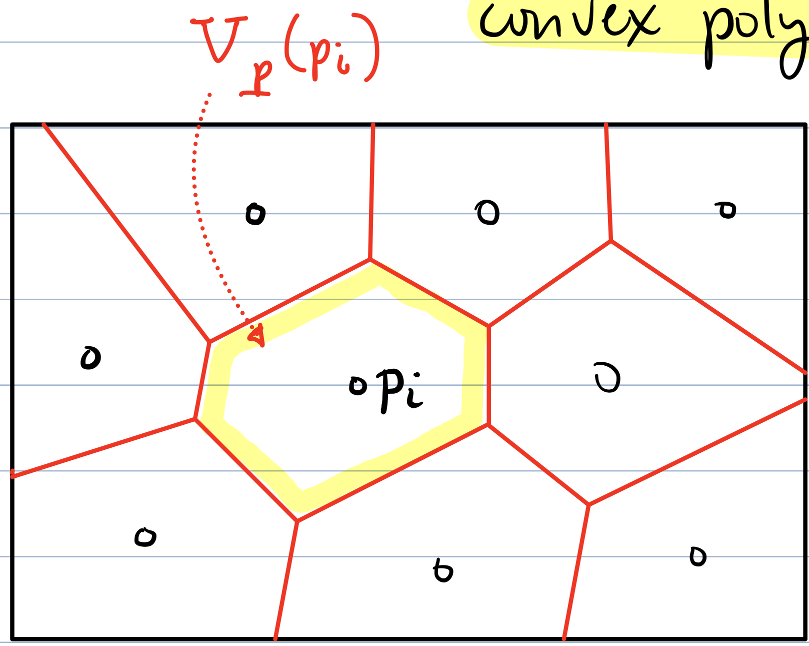
A fundamental structure for metric spaces.

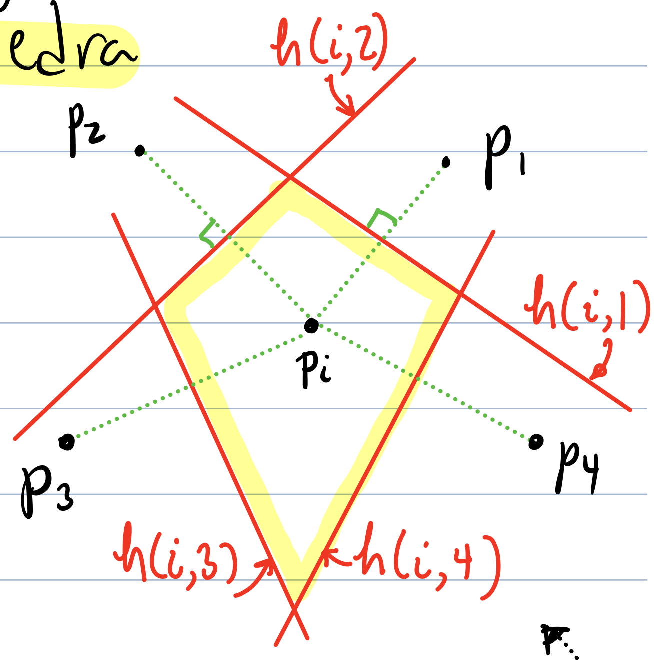
Given a point set $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^d$ called sites, we want to subdivide space based on each site's "region of influence"

Def: Voronoi cell for site p_i

$$V_P(p_i) = \{ q \in \mathbb{R}^d \mid \|p_i - q\| < \|p_j - q\|, \forall j \neq i \}$$

Obs: - Voronoi cells are disjoint
- For Euclidean dist, Voronoi cells are (possibly unbounded) convex polyhedra





Let $h(i,j) = \{ q \mid \|p_i - q\| < \|p_j - q\| \}$

$h(i,j)$ - halfspace bounded by perpendicular bisector between p_i + p_j

$$\text{Vor}(p_i) = \bigcap_{j \neq i} h(i,j)$$ – intersection of halfspaces ⇒ polytope

Def: Vor(P) is the subdivision (cell complex) induced by P's voronoi cells.

- Vor(P) covers $\mathbb{R}^d$
- Has n cells (faces of dim d)
- Polyhedral subdivision (for Euclidean dist)
- Combinatorial complexity:

  $\mathbb{R}^2$: $O(n)$ edges + vertices

  $\mathbb{R}^d$: $O(n^{\lceil d/2 \rceil})$ size [Closely related to convex polytopes in $\mathbb{R}^{d+1}$]

Many applications:

Nearest neighbor search:

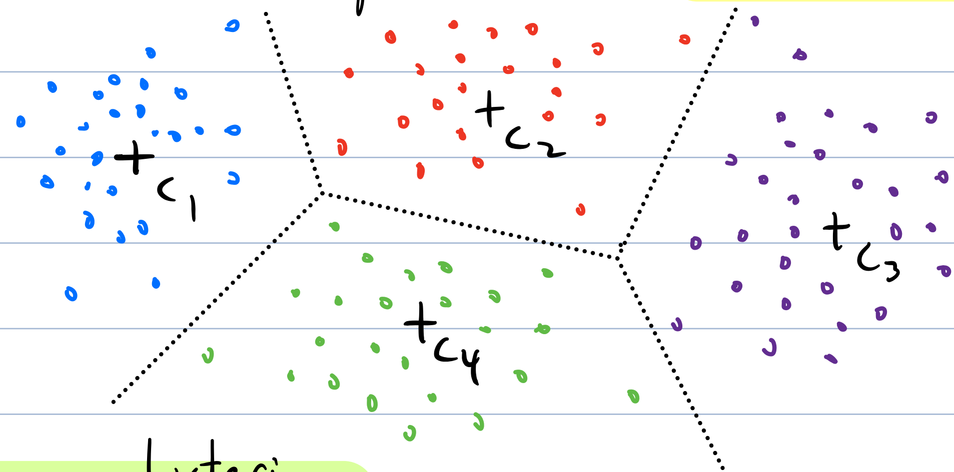
Preprocess a set of sites $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$ s.t. given any query point $q \in \mathbb{R}^d$ can find q's nearest site

How? - Compute Vor(P)
- Build a point-location data structure for Vor(P)

[Optimal in $\mathbb{R}^2$. Not as good in $\mathbb{R}^d$.]

Point-based Clustering:

- Given set T of training points, group them into k clusters
- Clusters are defined by k cluster centers $\{c_1, \ldots, c_k\}$
- Cluster membership based on closest center



- k-means clustering

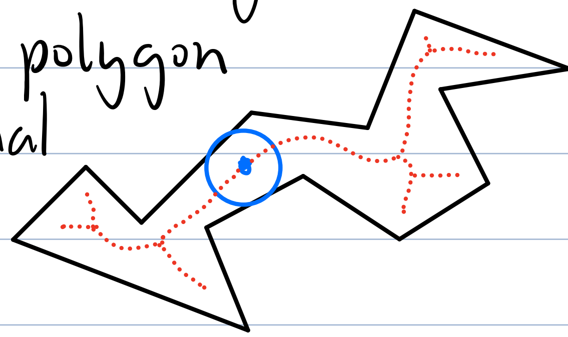
Variations:

- Other metrics: L_1-Vor diagram (Manhattan distance)
- Weighted pts:
  - Multiplicative: $dist(q, p_i) = \alpha_i \| p_i - q \|$
  - Additive: $dist(q, p_i) = \| p_i - q \| + \omega_i$
- k^{th} Nearest:
  - $Vor_k(P)$ = subdivide based on k^{th} closest

$Vor_n(P)$ = farthest point Vor. diag

- Other shapes:
  - Voronoi diagram of line segments
  - Medial axis of polygon

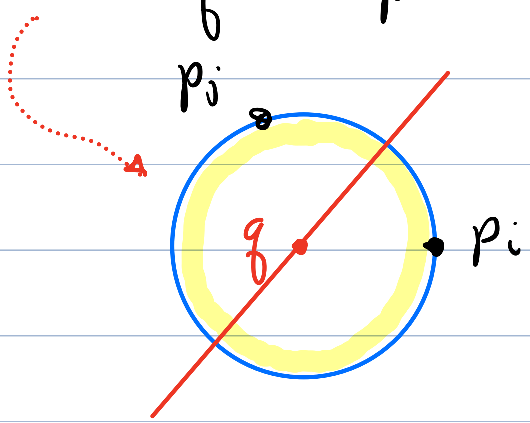
    centers of maximal disks

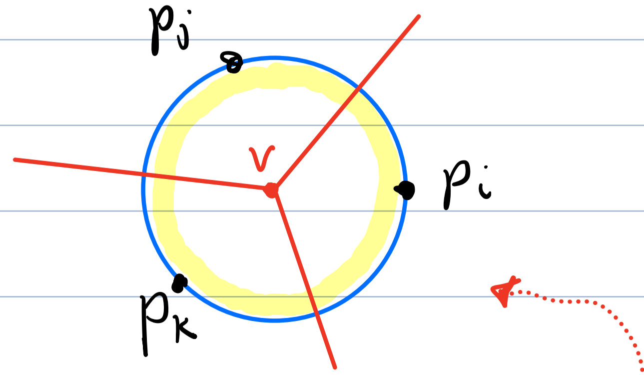


Properties of the Voronoi Diagram:

Empty-circle Property:

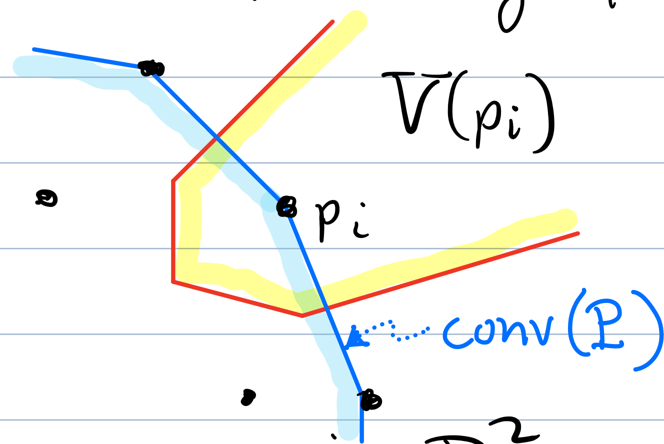
A pt q is on an edge of the Vor. diag iff there is a circle centered at q that passes through 2 sites + is otherwise empty.





Circumcircle Property: A pt v is a vertex of the diagram iff it is the center of a circle passing through 3 sites + is otherwise empty.

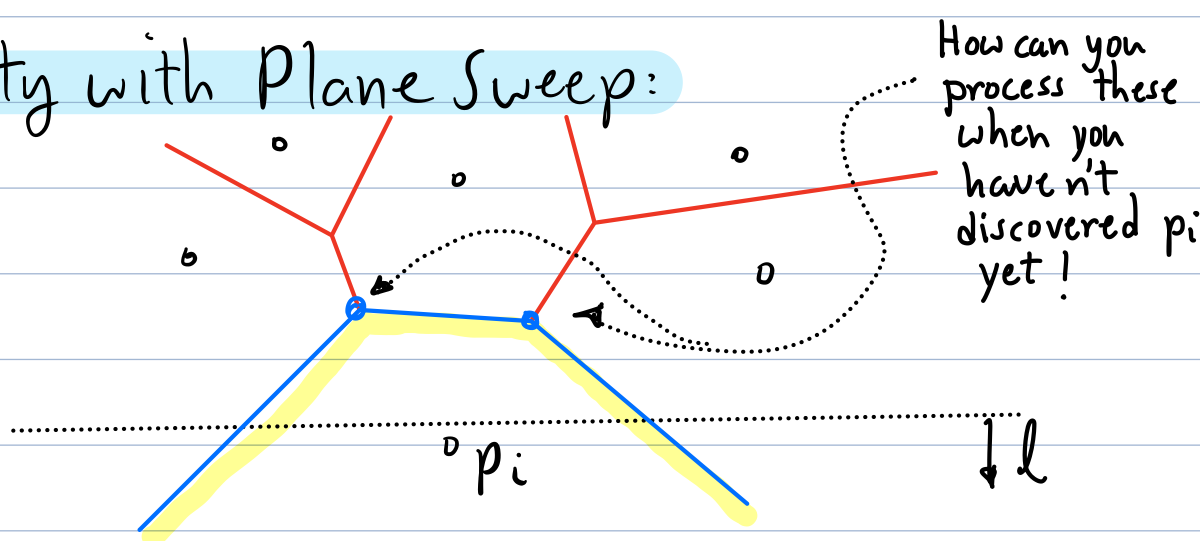
Hull Property: A site p_i has an unbounded Voronoi cell iff p_i is on boundary of convex hull of P.



Constructing Voronoi Diagrams in $\mathbb{R}^2$

- Incremental – add a site; update (best if randomized)
- Divide + Conquer - $O(n \log n)$
- Plane Sweep (this lecture)
  - Fortune's Algorithm - $O(n \log n)$

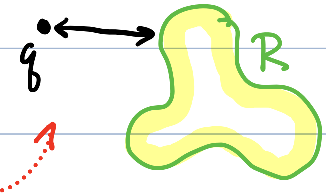
Difficulty with Plane Sweep:



Clever twist: We'll maintain two sweeping structures: sweep line + beach line

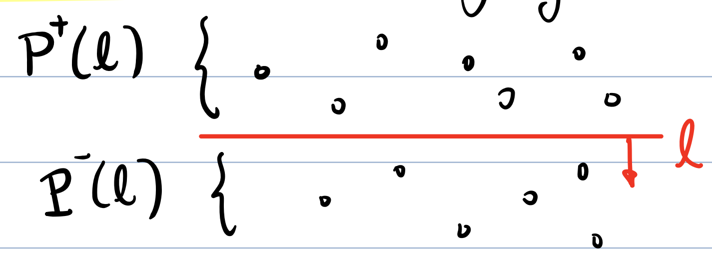
Def: Given a set of pts R and pt q, define

$$\mathrm{dist}(q, R) = \min_{p \in R} \|p - q\|$$



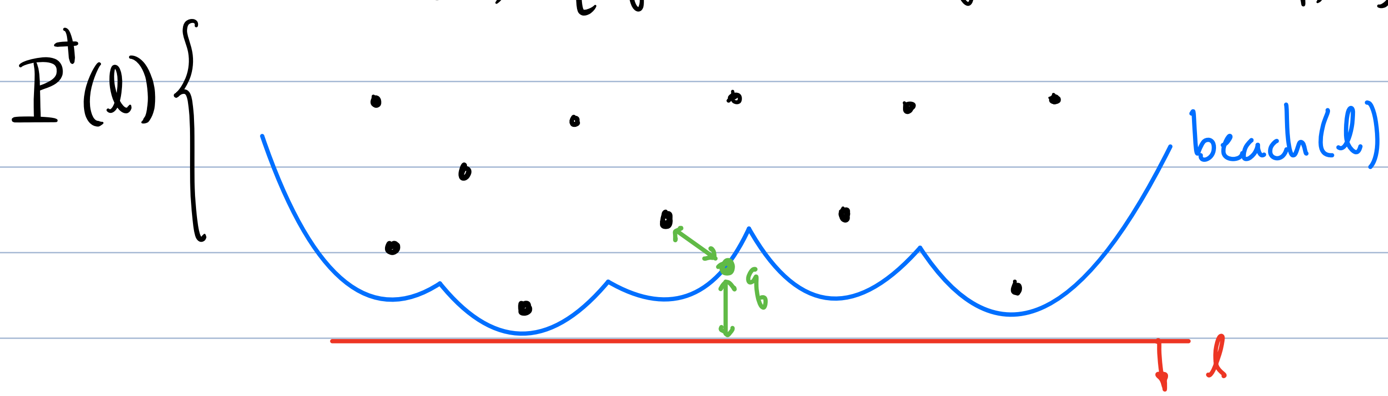
Given a sweep line ℓ (horizontal + moving down) define

$P^+(\ell)$ to be sites lying above ℓ



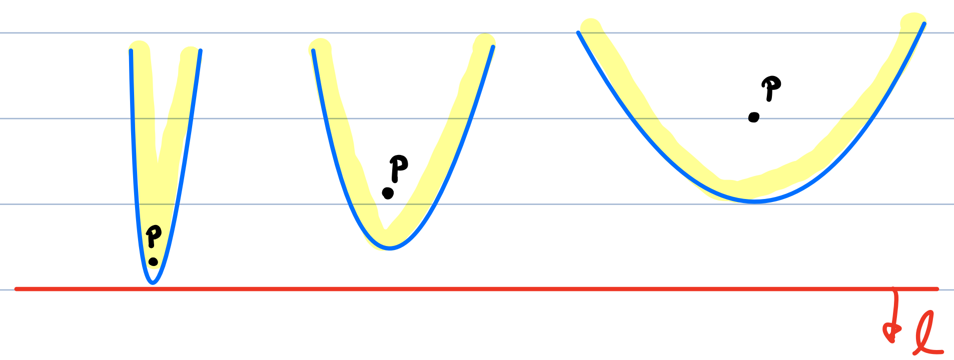
Given sweep line ℓ, define the beach line to be set of pts $q \in \mathbb{R}^2$ that are equidistant from $P^+(\ell)$ and ℓ

$$\mathrm{beach}(\ell) = \{ q \in \mathbb{R}^2 \mid \mathrm{dist}(q, P^+(\ell)) = \mathrm{dist}(q, \ell) \}$$



Beach-line Structure:

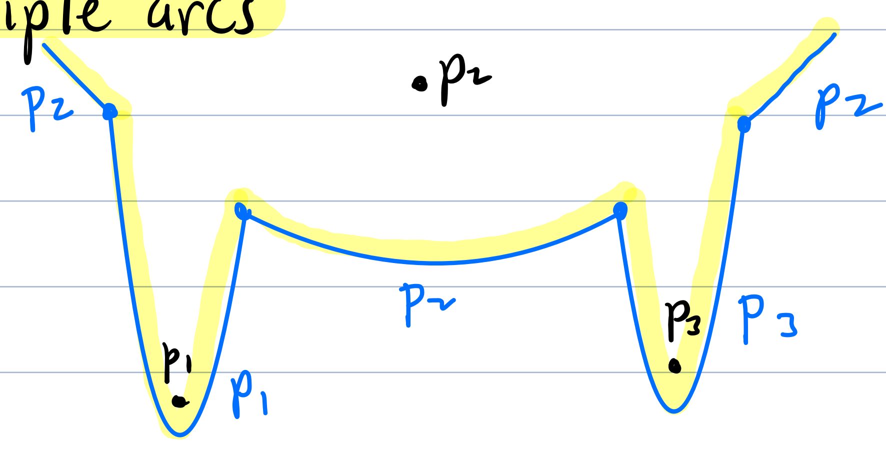
The points equidistant to a site p + line ℓ form a parabola



(wider as p is higher)

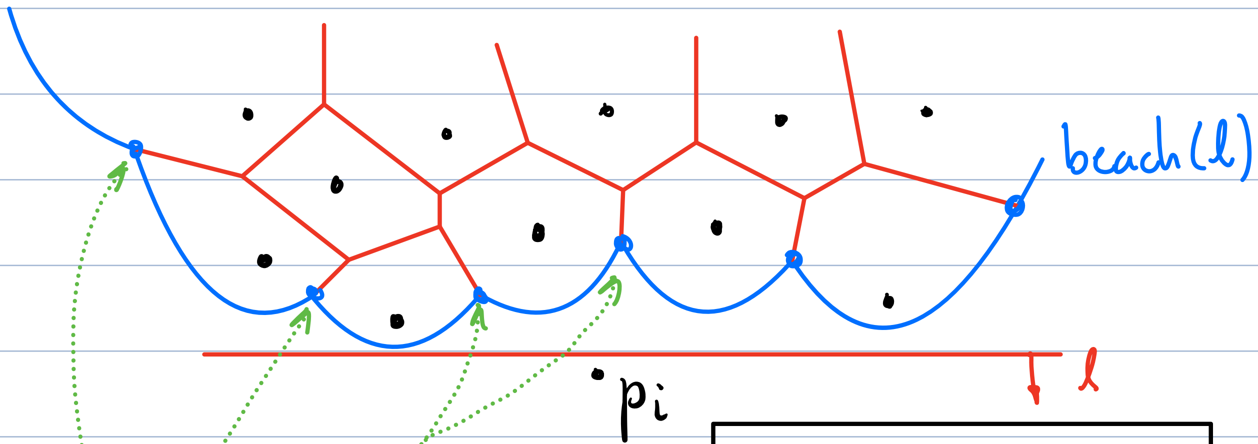
The beach line is the lower envelope of these parabolas for all sites in $P^+(\ell)$

- Beach line is x-monotone
- A single site may contribute 0, 1, or multiple arcs



- Total complexity is $O(|P^+(\ell)|) = O(n)$ [Proof: Exercise]

Key: The portion of $\mathrm{Vor}(P)$ above the beach line is "safe" from sites lying below ℓ.



Break points

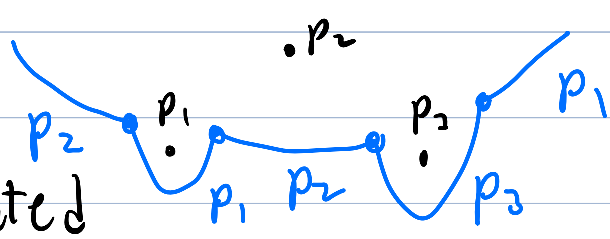
p_i cannot affect $\mathrm{Vor}(P)$ above beach(ℓ)

Fortune's Algorithm:

Sweep-line status:

- y-coord of sweep line
- seq. of sites (left to right) that contribute arc to beach line (e.g. $\langle 2, 1, 2, 3, 2 \rangle$)

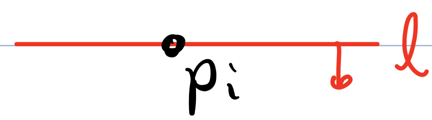
- Parabolic arcs not computed
- Breakpts generated as needed



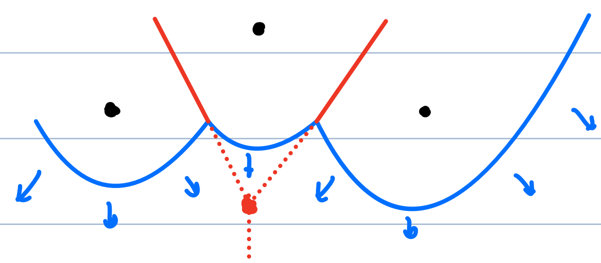
Voronoi diagram: Portion of Voronoi diagram (rep. as DCEL) above beach line is stored/updated.

Events:

Site event: Sweep line passes over a site



Vertex event (circle event):
- A new Voronoi vertex is discovered
≡ An arc on beach line vanishes

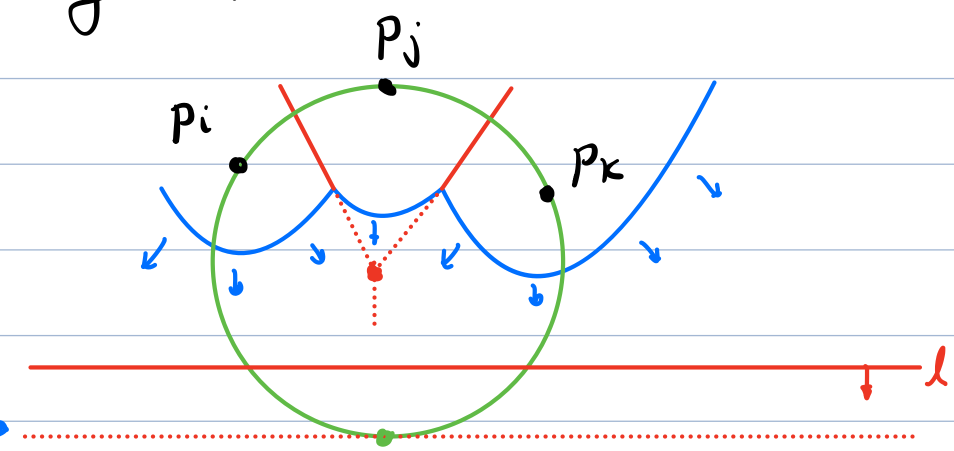


Priority Queue: Stores y-coords for sweep line at events.

Site events: Easy - just y-coord of site (static)

Vertex events: Tricky! (see below)

Scheduling vertex events:



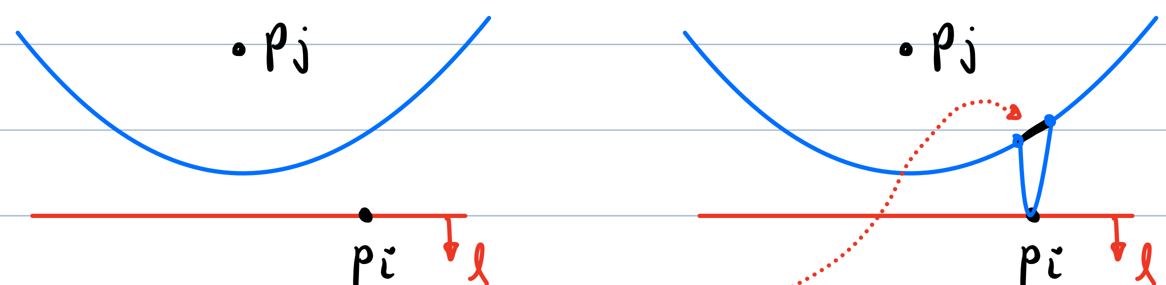
- For each consecutive triple $\langle \ldots p_i, p_j, p_k \ldots \rangle$ on beach line compute lowest y-coord of circumcircle (p_i, p_j, p_k)
- Schedule vertex event when sweep line reaches this y-coord.

Site Event: for site p_i

(1) Find arc of beach line above p_i

(2) Split this arc:

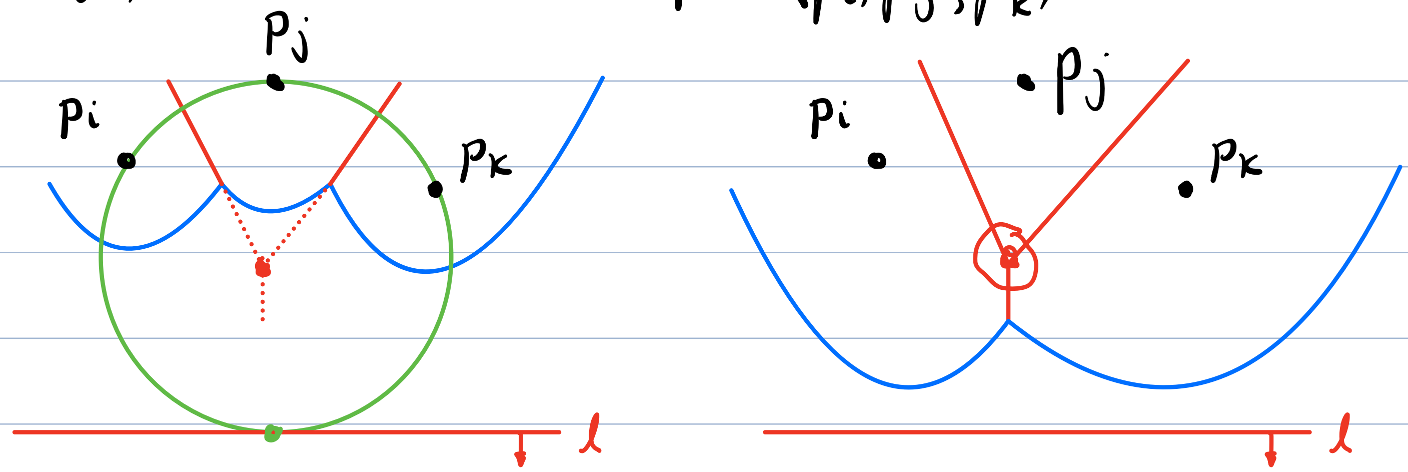
$$\langle \ldots, p_j, \ldots \rangle \Rightarrow \langle \ldots p_j, p_i, p_j, \ldots \rangle$$



(3) Create a new "dangling" edge (between p_i + p_j) add to Voronoi diagram.

(4) Update priority queue vertex events (below)

Vertex Event: for triple $\langle p_i, p_j, p_k \rangle$



(1) Delete p_j's arc from beach line

$\langle \cdots p_i \; p_j \; p_k \cdots \rangle \Rightarrow \langle \cdots p_i \; p_k \cdots \rangle$

(2) Create new Voronoi vertex joining edges $p_i p_j$ + $p_j p_k$ in diagram

(3) Start new (partial) Voronoi edge for $p_i p_k$

(4) Update priority queue vertex events

Analysis: - $O(n)$ events
- $O(\log n)$ per event
- $O(n \log n)$ total time