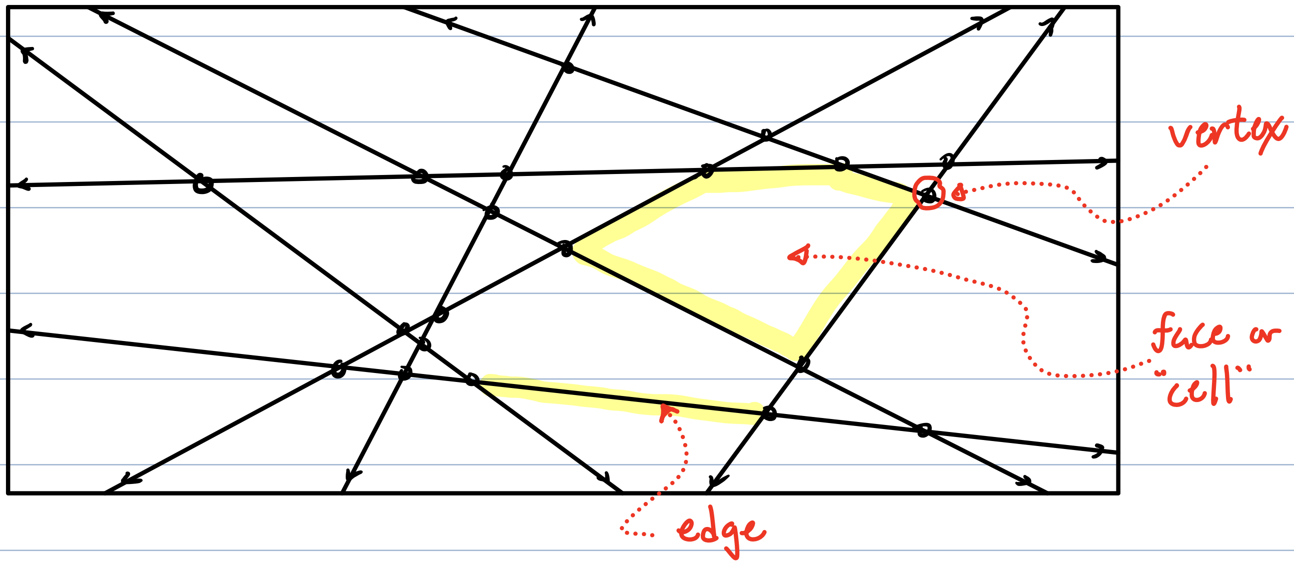
# CMSC 754 - Computational Geometry
## Lecture 13 - Line Arrangements

**Arrangement:**

Given a set $L = \{\ell_1, \dots, \ell_n\}$ of lines in $\mathbb{R}^2$ (generally $(d-1)$-dim hyperplanes in $\mathbb{R}^d$), they subdivide the plane into a cell complex called the arrangement of $L$, or $A(L)$.




**Combinatorial Properties:**

**Lemma:** Given $n$ lines $L$ in gen'l position in $\mathbb{R}^2$:

(i) $A(L)$ has $\binom{n}{2} = \frac{1}{2} \cdot n(n-1)$ vertices

(ii) $A(L)$ has $n^2$ edges

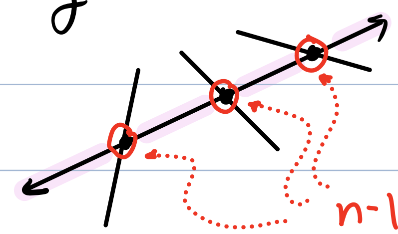
(iii) $A(L)$ has $\binom{n}{2} + n + 1 = \frac{1}{2}(n^2+n+2)$ cells

Proof:

(i) Each pair intersects once $= \binom{n}{2}$ ✓

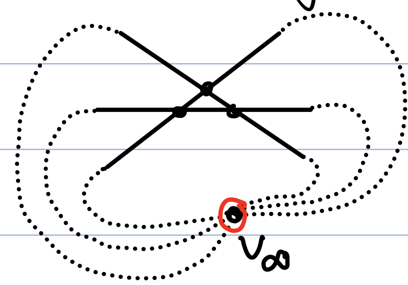
(ii) Each line is split by $n-1$ others into $n$ edges

$\Rightarrow$ $n^2$ total ✓




(iii) Add a vertex at $\infty$ of degree $n$ to tie off all unbounded edges

$$v = \binom{n}{2} + 1$$
$$e = n^2$$




By Euler's formula:

$$v - e + f = 2$$
$$\Rightarrow \left(\binom{n}{2}+1\right) - n^2 + f = 2$$
$$\Rightarrow f = 2 + n^2 - \left(\binom{n}{2}+1\right)$$
$$\Rightarrow f = 2 + n^2 - \frac{n(n-1)}{2} - 1$$
$$= \frac{1}{2}(n^2 + n + 2) \quad ✓ \qquad \square$$

[In $\mathbb{R}^d$, complexity is $\Theta(n^d)$]

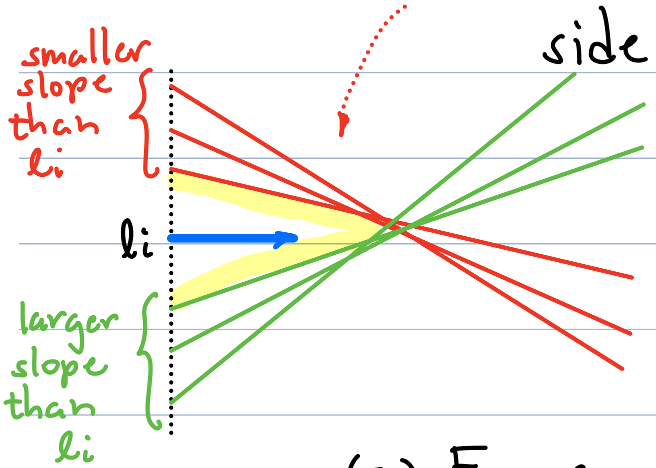
**Incremental Construction:** (not randomized)

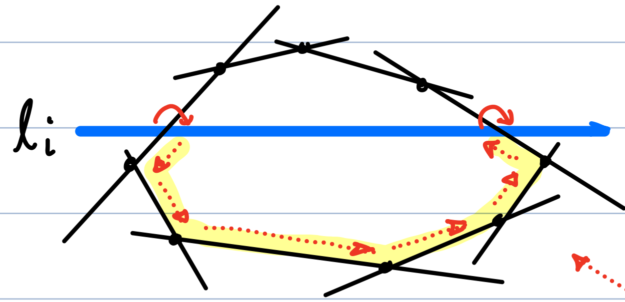
**Idea:** Add lines one by one (in any order)
Update the structure after each

**Notation:** $L_i = \{\ell_1, \ldots, \ell_i\}$

## How to add the $i^{th}$ line? $\ell_i$

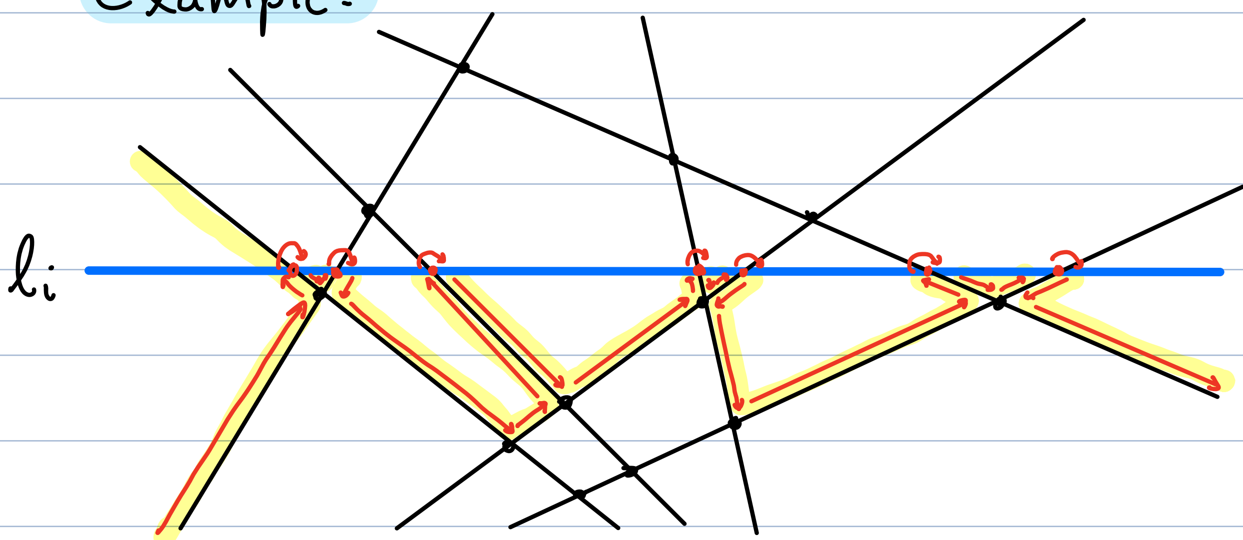
(1) Find the unbounded cell on left side where $\ell_i$ starts (slope based)




(2) For each face of $A(L_{i-1})$ that intersects $\ell_i$, walk along its lower boundary to determine where it exits this cell

Example:




- Once we know entry-exit points on each face - we update arrangement in $O(i)$ time (DCEL)
- How long to crawl around edges?

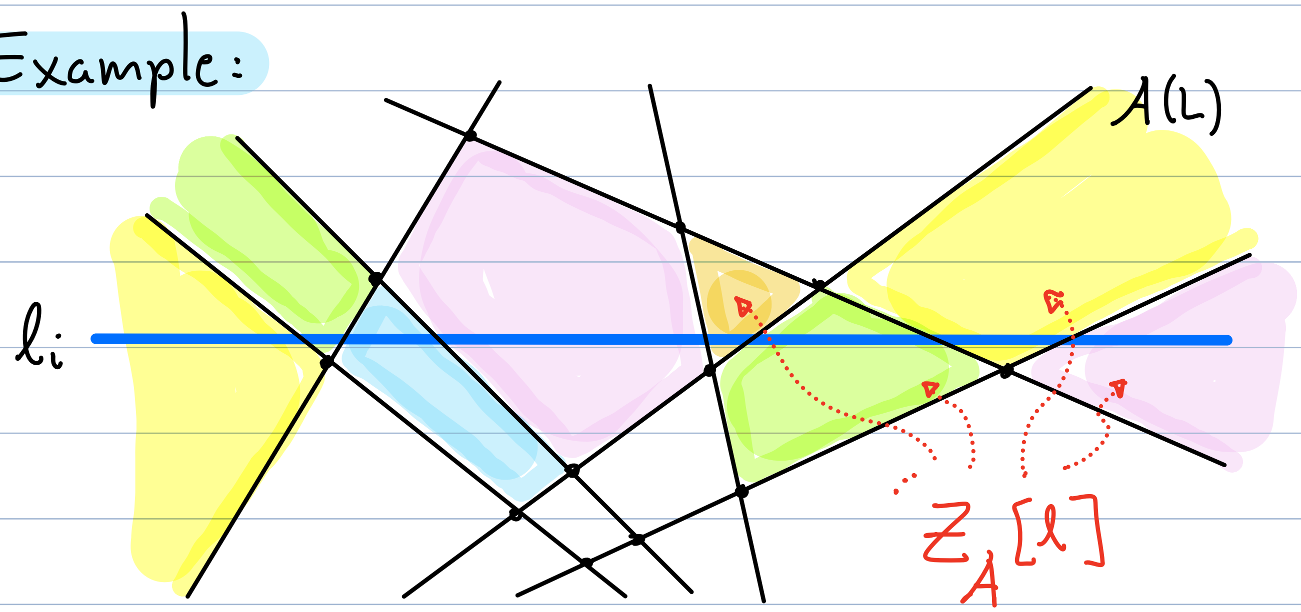
**Naive analysis:** On adding $\ell_i$

- $\ell_i$ crosses $i$ cells
- each cell may have as many as $i-1$ edges
- crawl takes $O(i(i-1)) = O(i^2)$ time
- total time $\cong \sum_{i=1}^{n} i^2 = O(n^3)$

Can it really be this bad?

**Zone:** Given an arrangement $A = A(L)$ and a line $\ell \notin L$, zone of $\ell$ in $A$, $Z_A(\ell)$ is the set of cells of $A$ that $\ell$ intersects.

**Example:**



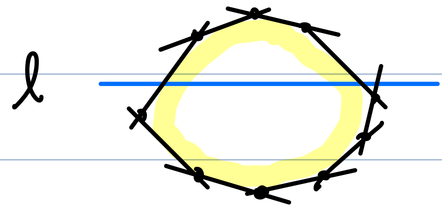

**Obs:** Crawl time $\leq$ no. of edges on the zone of $\ell_i$ in $A(L_{i-1})$ $[Z_{A(L_{i-1})}(\ell_i)]$

$\rightarrow$ We'll show this is $O(i)$ not $O(i^2)$

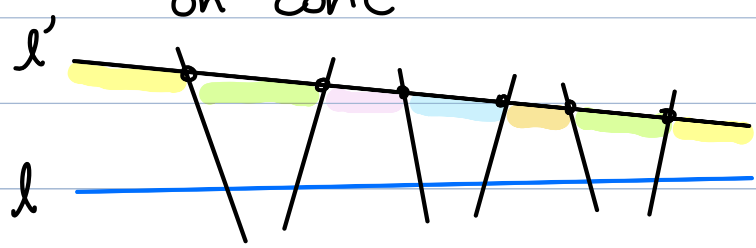
**Theorem: (Zone Theorem)** Given an arrangement $\mathcal{A}(L)$ where $|L| = n$ and any line $\ell \notin L$, the number of edges in $Z_{\mathcal{A}}(\ell) \leq 6n$

**How to prove this?**

cell by cell? Some cells have high complexity




line by line? Some lines appear many times on zone



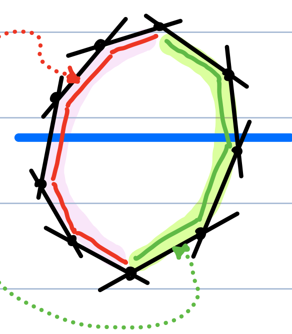

**Our approach:**

- Partition edges of zone into two classes (left side + right side)
- Show (by induction) at most $3n$ of each

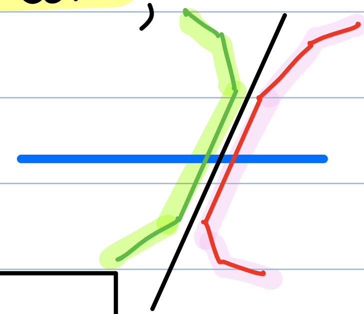
A zone edge is:

left bounding: on left side of cell

right bounding: on right side of cell



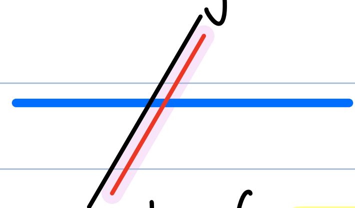
**Note:** Some edges appear twice in the zone, both as left/right bounding



**Claim:** At most $3n$ left-bounding edges.

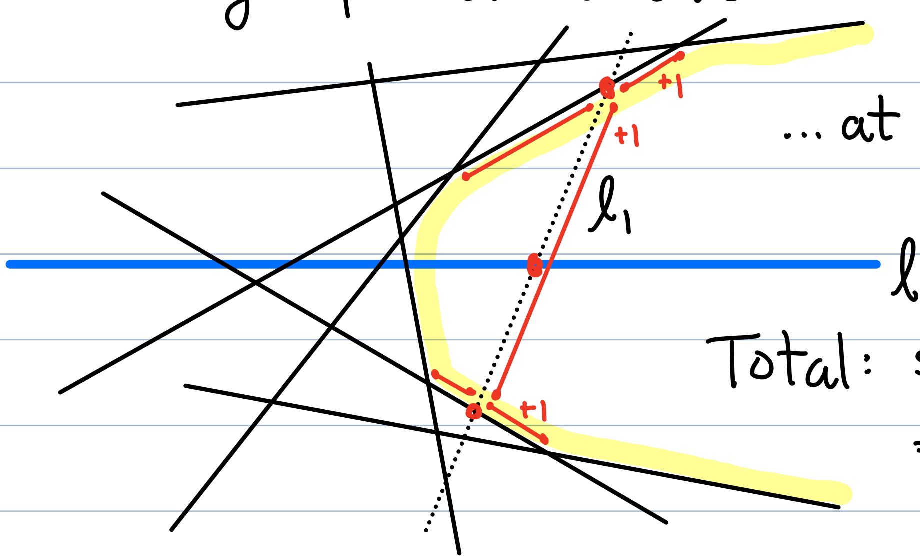
**Proof:** By induction on $n$

$n=1$: Just one LB edge $1 \le 3 \cdot 1$ ✓



$n \geq 2$: I.H. arrangement of $n-1$ lines has $\le 3(n-1)$ LB edges in zone

- Let $\ell_1 \in L$ be rightmost line to cross $\ell$
- Removing $\ell_1 \Rightarrow$ at most $3(n-1)$ LB edges
- Adding $\ell_1$ back creates ...




... at most 3 new LB edges

Total: $\le 3(n-1)+3$
$= 3n$ $\square$

**Thm:** Given a set $L$ of $n$ lines in $\mathbb{R}^2$, $A(L)$ can be built in time $O(n^2)$ [and has size $O(n^2)$ ... so this is optimal]

**Proof:**
- Apply incremental construction
- Inserting $\ell_i$ takes time $\sim$ no. of edges in $Z_{A(L_{i-1})}(\ell_i) \leq 6(i-1)$
- Total time $\leq \sum_{i=1}^{n} 6(i-1) = 6 \sum_{i=0}^{n-1} i = O(n^2)$

**Applications:**

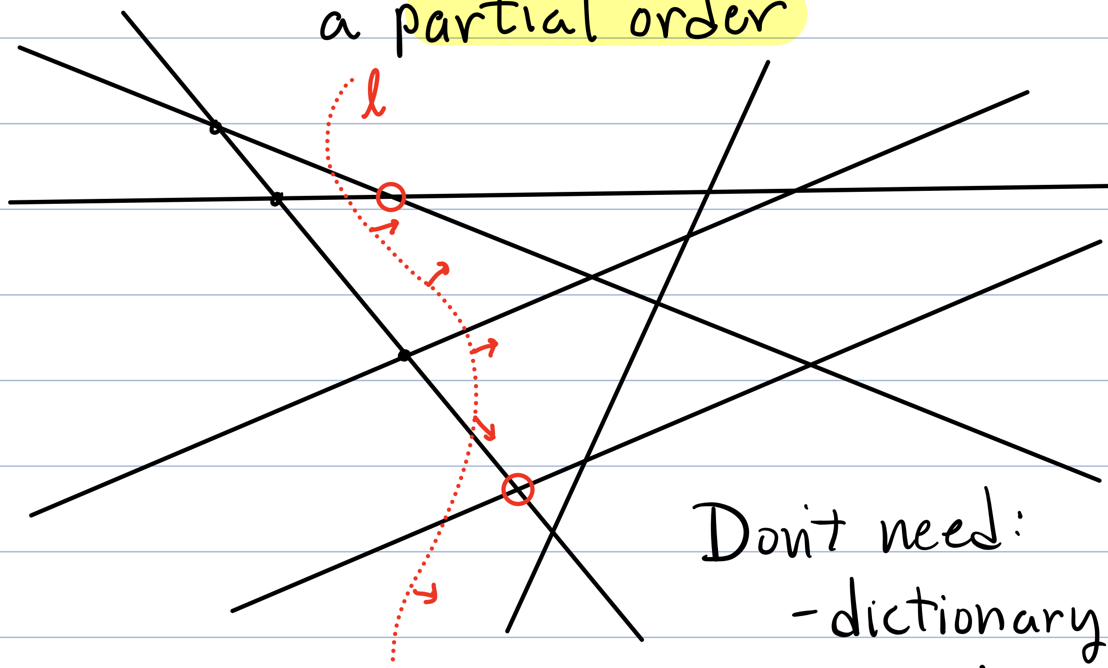
Line arrangements can be used to solve many problems
- mostly $O(n^2)$ time
- often using duality

How to process an arrangement?
- Build it + traverse it like a graph
  $O(n^2)$ time, $O(n^2)$ space
- Plane sweep
  $O(n^2 \log n)$ time, $O(n)$ space
- Topological plane sweep
  $O(n^2)$ time, $O(n)$ space

  Not covered, but applicable pretty much whenever plane sweep is. ← you may assume this

## Topological plane sweep:

- A relaxed version of plane sweep
- Vertices are not swept in strict left to right order, but based on a partial order




Don't need:

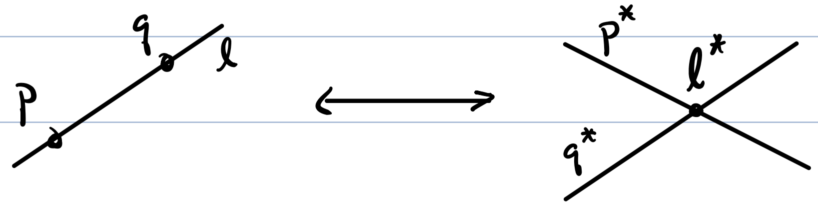
- dictionary
- priority queue

$\Rightarrow$ saves $\log n$ factor

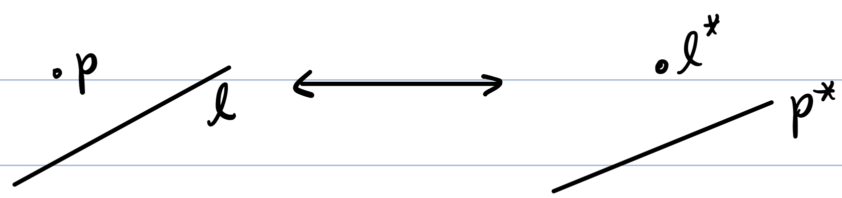
## Recall: Dual transformation

$p = (a, b) \longleftrightarrow p^*: y = ax - b$

$\ell: y = ax - b \longleftrightarrow \ell^*: (a, b)$



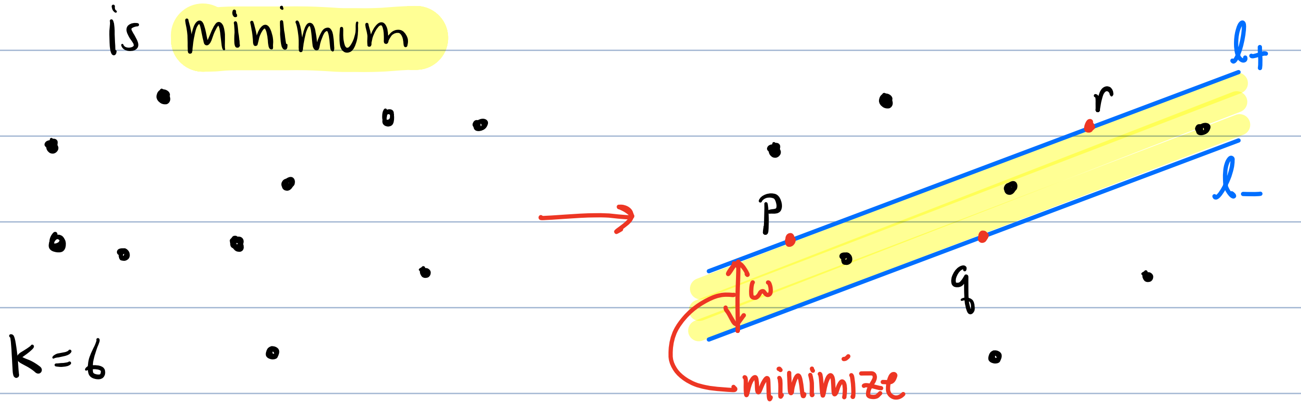

incidence preserving




order reversing

## Narrowest k-corridor:

- Given a set $P=\{p_1,\dots,p_n\}$ in $\mathbb{R}^2$ and integer $3 \le k \le n$, find pair of parallel (non vertical) lines that enclose k pts so that vertical distance between lines is minimum




## Primal form:

- Let $\ell_+$ + $\ell_-$ be upper + lower lines of "slab"

parallel ⇒ same slope

$$\ell_+: y = ax - b_+ \qquad b_+ \le b_-$$
$$\ell_-: y = ax - b_-$$

order reversed due to negation

- Vertical width: $w = b_- - b_+$
- k pts of P lie on or between $\ell_-$ + $\ell_+$

## Local optimality:

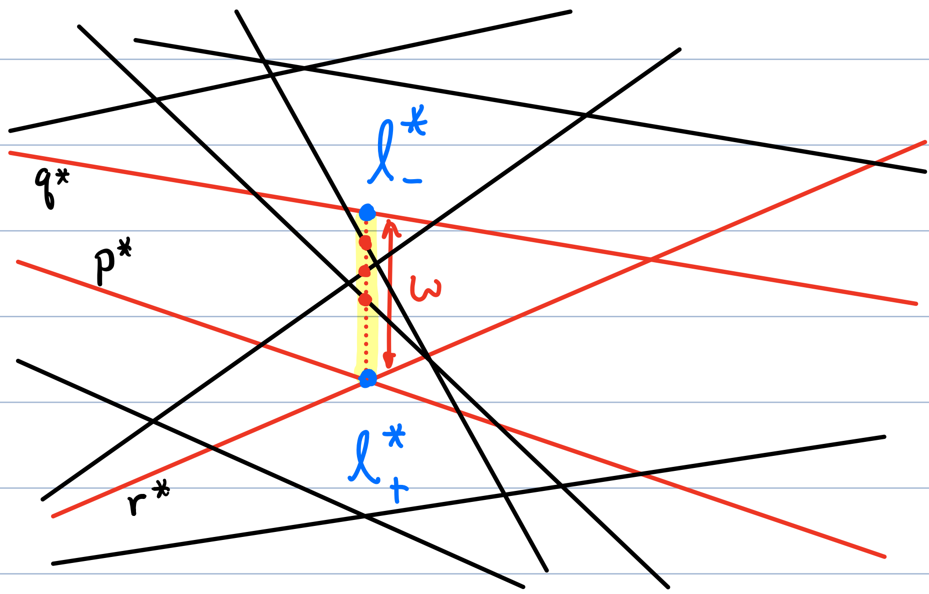
3 pts of P will lie on $\ell_+$ + $\ell_-$, 2 on one edge + 1 on other

- If 0, 1, or 2 can make width smaller
- If 4 or more – not gen'l position

vertical line segment

## Dual form:

- $\ell_+^* + \ell_-^*$ are pts $(a, b_+) + (a, b_-)$
- vertical distance $b_- - b_+$
- k lines of $P^*$ pass through or between these pts




## Local optimality:

3 lines will pass through $\ell_-^* + \ell_+^*$ with 2 on one side + one on other

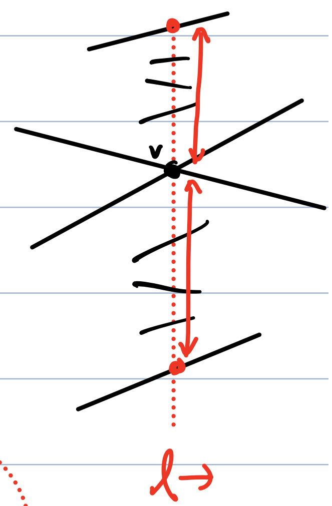
## Narrowest-Corridor(P, k):

(1) $P^* \leftarrow$ dual lines of $P$

(2) Plane sweep through $P^*$.

(3) On arriving at each vertex v, compute vertical distance to lines k-2 above + k-2 below

(4) Return smallest such distance



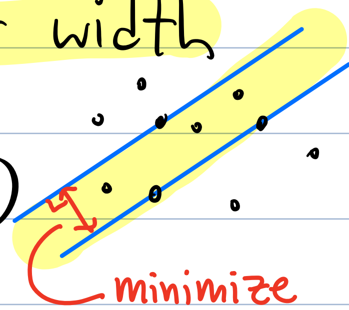
**Correctness:** (Argued above)

**Time:** $O(n^2 \log n)$ time + $O(n)$ space

Can access in $O(1)$ time since sweep line can be stored in array

↳ can reduce to $O(n^2)$ by topol. plane sweep.

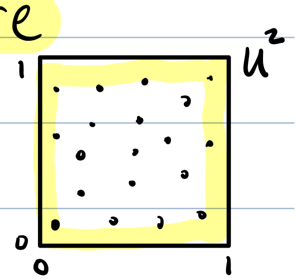
**Aside:** It is easy to generalize this to minimize perpendicular width (Just apply a correction factor when computing widths)




## Halfplane Discrepancy:

Let $U^2 = [0,1]^2$ denote the unit square

Given $n$ pts $P = \{p_1, \dots, p_n\} \subset U^2$, how close is $P$ to being uniformly distributed over $U^2$?



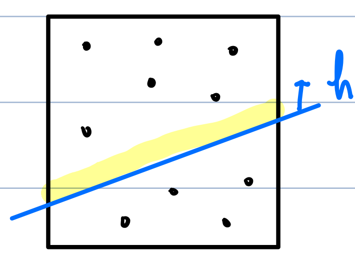

**Idea:**

For any halfplane $h$, let

$\mu(h) = \text{area}(h \cap U^2)$ $[0 \leq \mu(h) \leq 1]$

the fraction of $P$ in $h$

$\mu_P(h) = |h \cap P| / |P|$ $[0 \leq \mu_P(h) \leq 1]$




$\mu(h) = 2/3 = 0.666\ldots$

$\mu_P(h) = 6/10 = 0.6$

If $P$ is uniformly distrib., we expect

$$\mu(h) \approx \mu_P(h) \quad \forall h$$

To measure how uniform is $P$, define:

$$\Delta(P) = \max_h |\mu(h) - \mu_P(h)|$$

Called the halfplane discrepancy of $P$ $\quad [0 < \Delta(P) \leq 1]$

can't be perfect

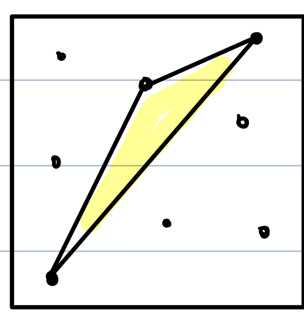
## Questions:

- Given $P \subset U^2$, what is $\Delta(P)$?
- How low can $\Delta(P)$ be for any set of size $n$?
- How to generate optimally uniform set $P_{OPT}$ of a given size $n$? ($\Delta(P_{OPT})$ is min. possible)
- Other measures of discrepancy?
  - Triangle discrepancy
  - Heilbronn's Triangle Problem:

    Given any set of $n$ pts $P$ in $U^2$, how large can the min area triangle be?

    Conj: $O(1/n^2)$

    Open for a century!



## Computing $\Delta(P)$ for a set $P \subset U^2$.

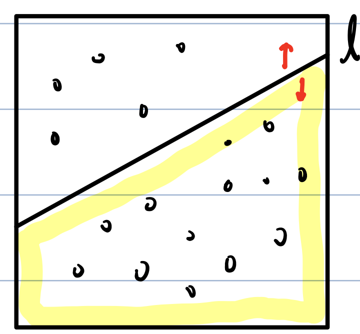
**Key:** Identify $O(n^2)$ candidates for halfplane that maximizes discrepancy.
- Compute discrepancy for each
- Return the max

**Lemma:** Given pt set $P$, let $h$ be halfplane of max discrepancy. Let $\ell$ be $h$'s bounding line. Either:
(i) $\ell$ passes through pt $p_i \in P$, and $p_i$ is midpoint of $\ell \cap U^2$
(ii) $\ell$ passes through two pts of $P$.

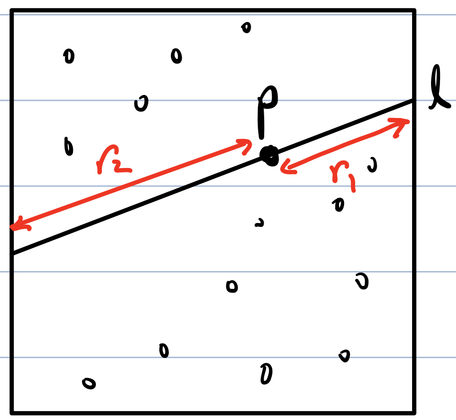
**Proof:**

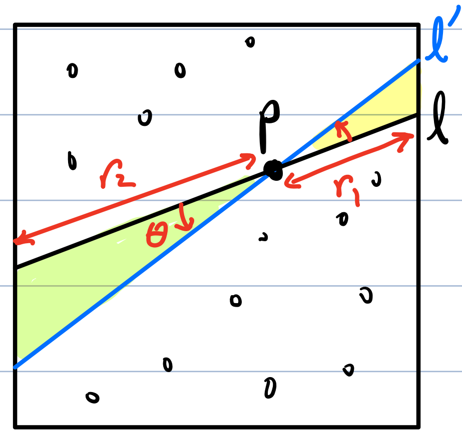
**Approach:** Consider any line $\ell$. We'll show unless it satisfies (i) or (ii) we can perturb it to increase discrepancy.

**Case 1:** $\ell$ passes through no pt of $P$ – perturbing $\ell$ up or down increases discrepancy.



**Case 2:** $\ell$ passes through a pt $p \in P$, but $p$ is not midpt of $\ell \cap U^2$



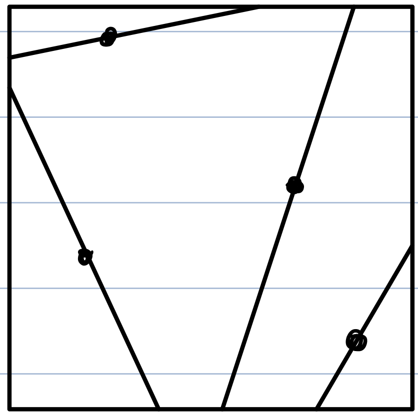

$p$ splits $\ell \cap U^2$ into two segments of lengths $r_1 + r_2$. Since $p$ is not midpt, may assume w.l.o.g. $r_2 > r_1$

If we rotate $\ell$ by small angle $\theta$ about $p$ we increase/decrease area by $\sim$

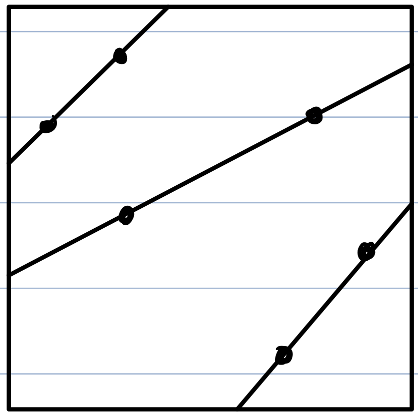
$$r_2^2 \cdot \theta - r_1^2 \theta = (r_2^2 - r_1^2)\theta > 0$$

Some small rotation will increase discrepancy. $\square$

Type (i)



Type (ii)



Computing $\Delta(P)$:

Type (i):

- for each $p_i \in P$, compute lines $\ell$ s.t. $p_i$ on midpt of $\ell \cap U^2$
  - Count no. of pts on either side of $\ell$

$\rightarrow$ $n$ pts ; $O(1)$ lines each ; $O(n)$ time to count $\Rightarrow O(n^2)$ time

Type (ii):

- Dualize $P$ to $P^*$
- Perform plane sweep of arrangement $\mathcal{A}(P^*)$
- For each vertex of arrangement maintain no. of lines above + below on sweep line
- Compute discrepancy in $O(1)$ time for each vertex

$\rightarrow O(n^2)$ vertices

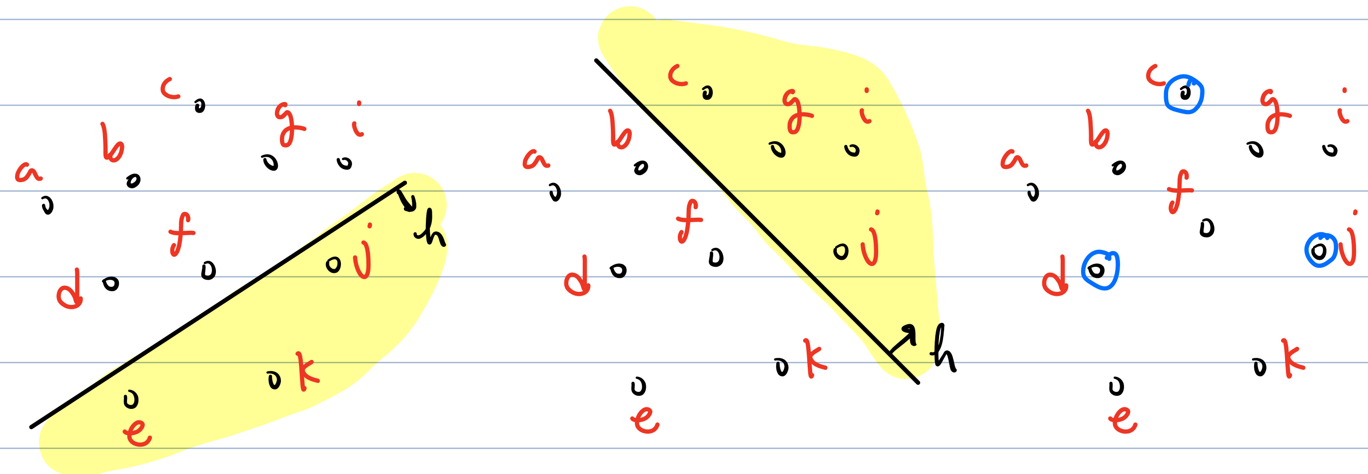
Can maintain counts in $O(1)$ time

$\Rightarrow O(n^2 \log n)$ time + $O(n)$ space

$O(n^2)$ by topol plane sweep

## Computing k-sets:

Given a set $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^2$ and integer $k$, $1 \le k \le n-1$, a k-set is a k-element subset of $P$ of the form $P \cap h$, for some halfplane $h$.




$\{e, k, j\}$ is a 3-set

$\{c, g, i, j\}$ is a 4-set

$\{c, d, j\}$ is not a 3-set

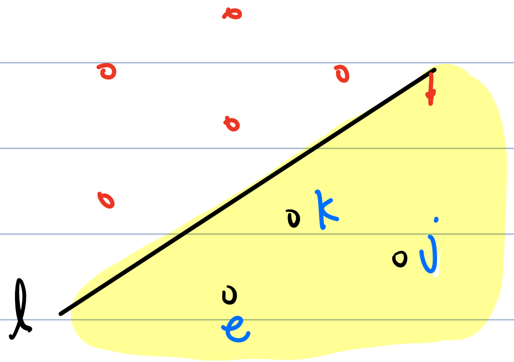
**Problem:** Given $P$ and $k$, enumerate all k-sets of $P$.

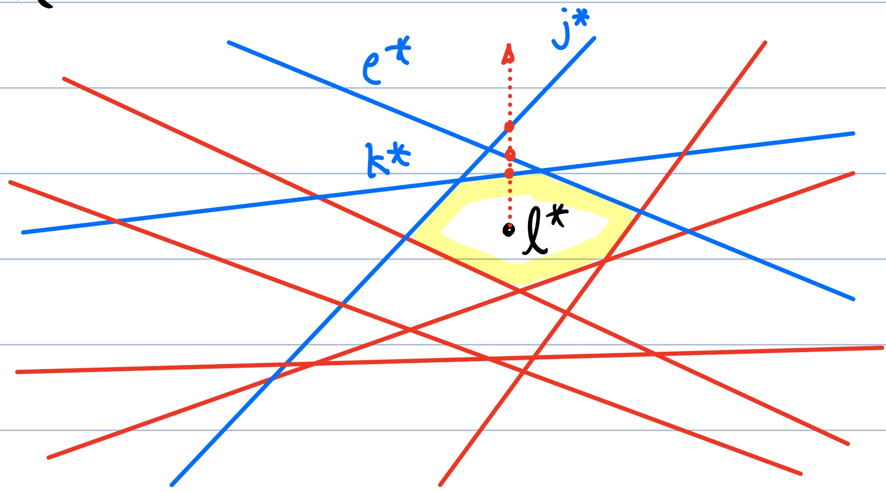
How many? Naive $\le \binom{n}{k} = O(n^k)$

Better $\le \binom{n}{2}$ (see below)

Best theoretic bounds: $O(n \log k) \dots O(nk^{1/3})$

# Dual equivalent?



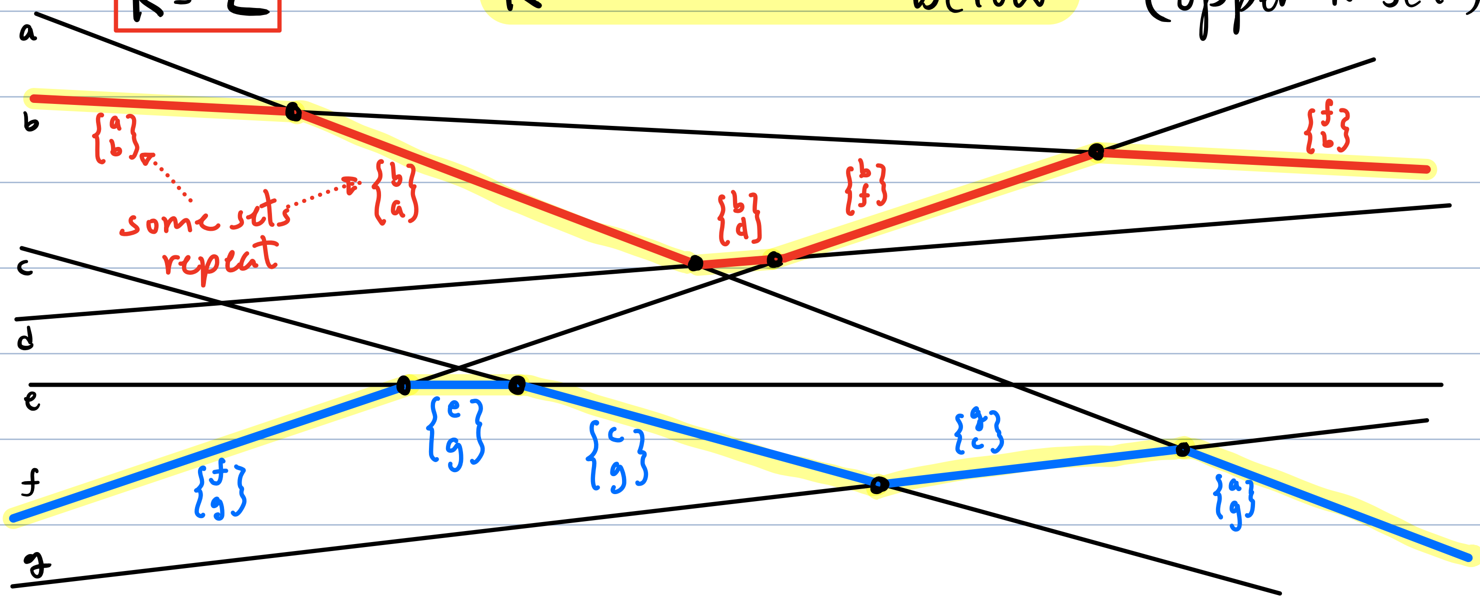

## By order reversal:

k-pts of $P$ lie below $\ell$ $\iff$ k-lines of $P^*$ pass above $\ell^*$

## Approach:

- Traverse the arrangement $A(P^*)$
- Identify all edges with
  - k lines on or above (lower k-set)
  - k " " " below (upper k-set)

$n = 7$
$k = 2$



**Level:** Given an arrangement of $n$ lines $A(L)$, for $1 \le k \le n$, define level $k$, $\mathcal{L}_k$, to be set of pts in $A(L)$ with

$\le k-1$ lines (strictly) above
$\le n-k$ lines (strictly) below

In above fi ure, we have shown $\mathcal{L}_2$ and $\mathcal{L}_6$
g

**Obs:** By applying plane sweep through $A(L)$, we can construct all levels in time $O(n^2)$

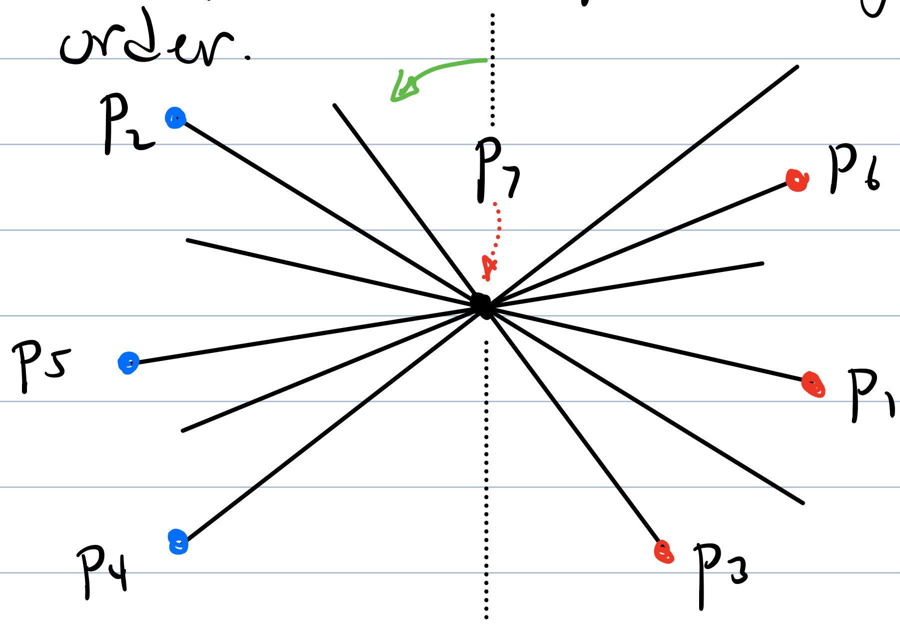
$\Rightarrow$ Can identify all $k$-sets of $P$ in time $O(n^2)$ by sweeping $A(P^*)$ + extracting levels $\mathcal{L}_k$ + $\mathcal{L}_{n-k+1}$

**Note:** To actually list the sets adds addional $k$ factor, total $O(k \cdot n^2)$

**Avoid duplicates?** Exercise

## Sorting angular sequences:

Given a set $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^2$, for each $p_i$, sort the remaining $n-1$ pts around $p_i$ in angular order.




$$\begin{bmatrix} 2 \\ 5 \\ 4 \\ 3 \\ 1 \\ 6 \end{bmatrix}$$

**Naive:** $O(n(n \log n)) = O(n^2 \log n)$
Sort angles for each point

**Better:** $O(n^2)$ using arrangements.

[See lect. notes for details]