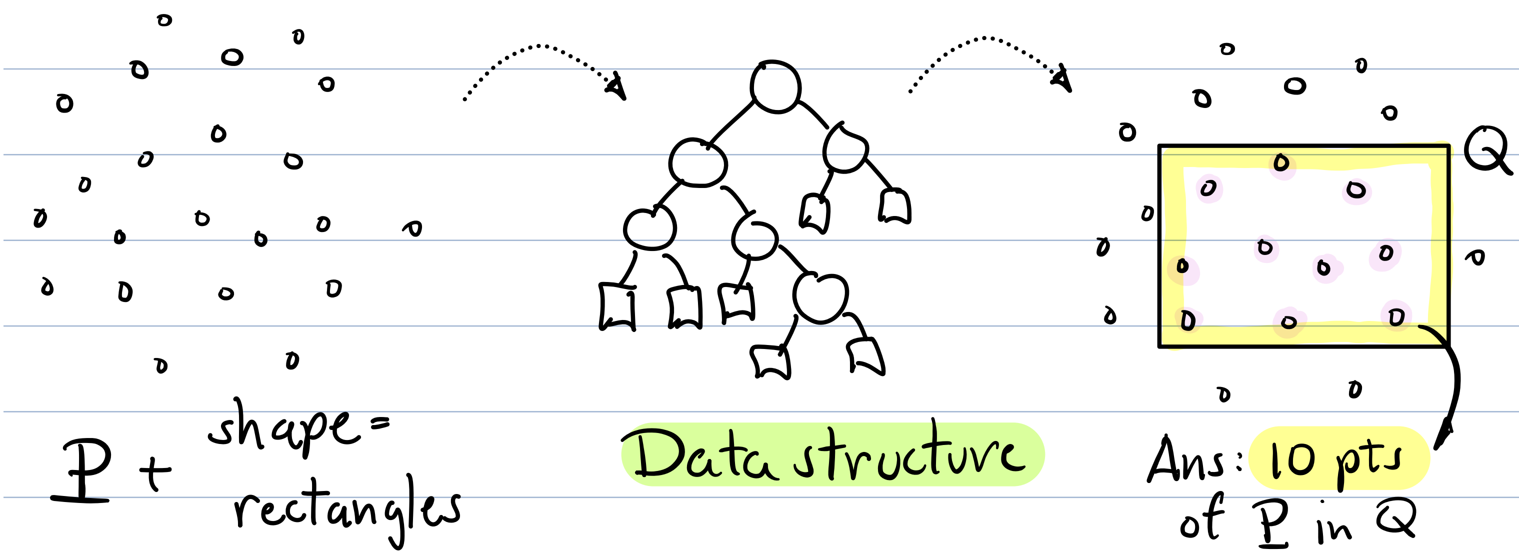
CMSC 754 - Computational Geometry
Lecture 15 - Orthogonal Range Trees

Recall: Range Search

Given a set of n pts $P = \{p_1, \ldots, p_d\} \subseteq \mathbb{R}^d$, and class of shapes (range space) preprocess P to answer range queries: Given shape Q, count/report the pts in $P \cap Q$.



Last lecture: kd-trees

$O(n)$ space / $O(n \log n)$ build time
$O(\sqrt{n})$ query time (in $\mathbb{R}^2$)
$O(n^{1-1/d})$ in $\mathbb{R}^d$

Today: Orthogonal Range Trees + Layered Data Structures

Multi-Layered Structures:

Suppose your ranges are formed from composing multiple (independent) queries:

Eg. Find all patients of

- age between 25..35 : Q_1
- weight ≤ 200 lbs : Q_2
- blood pressure ≥ 100 : Q_3

Idea: Design a data structure for each query type + "merge them"

How to merge?

- Build range structure for age for P

  $\Rightarrow$ Canonical subsets: $P_1, P_2, \ldots, P_m$

- For each P_i, build a range structure for weight

  $\Rightarrow$ Canonical subsets: $P_{i1}, P_{i2}, \ldots$

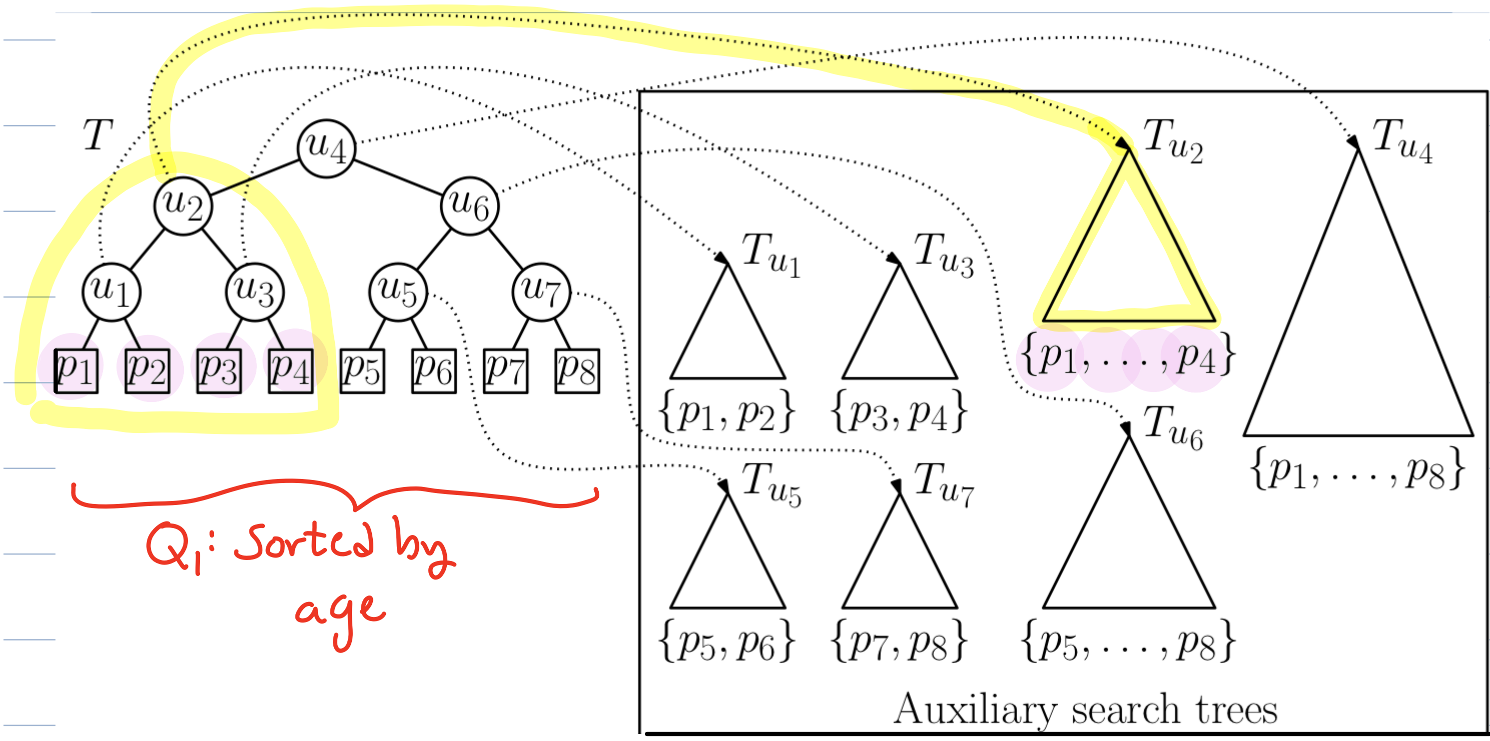
- For each P_{ij}, build range structure for blood pressure

⋮

Multi-Layered Search Tree:

- Store data in leaves of tree
- Each node's canonical subset consist of it's leaves
- For each node, build a search tree for its canonical subset
  - called its auxiliary tree

Example:

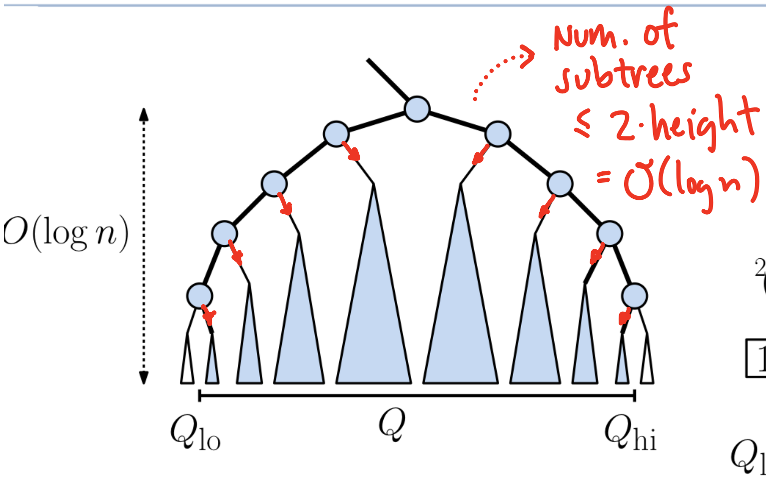


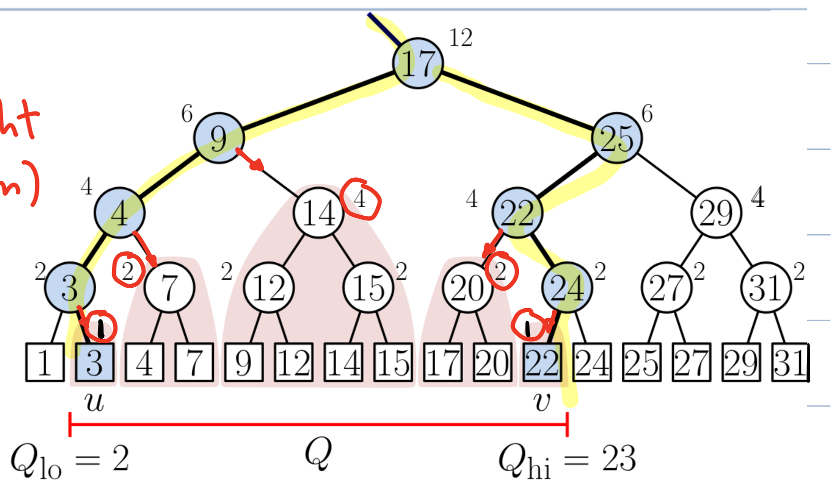
Q_1: Sorted by age

Q_2: Sorted by weight

1-Dimensional Range Tree: (Review)

- Given set of scalars: $\underline{P} = \{p_1, \dots, p_n\} \subseteq \mathbb{R}$
- Store as leaves in balanced search tree $\rightarrow O(n)$ space
  $\rightarrow O(n \log n)$ construct. time
- Each node u stores num. of leaves: $u.size$
- Given query interval $Q = [Q_{lo}, Q_{hi}]$
  - Identify $O(\log n)$ maximal subtrees that cover Q
  - Add up sizes for all these nodes





Query answer $= 1 + 2 + 4 + 2 + 1 = 10$

Range counting algorithm:

Node u :

- u.point : point p_i (if u is leaf)
- u.x : split value (if u internal)
- u.size : # leaves (if u internal)
- u.left, u.right : children

```
rangeIDx(Node u, Range Q, Interval C = [x0, x1])
    if (u is leaf)
        return { 1 if u.point ∈ Q
               { 0 o.w.
    else if (C ∩ Q = ∅)            (no overlap)
        return 0
    else if (C ⊆ Q)                (contained)
        return u.size
    else                           (recurse)
        return rangeIDx(u.left, Q, [x0, u.x])
             + rangeIDx(u.right, Q, [u.x, x1])
```

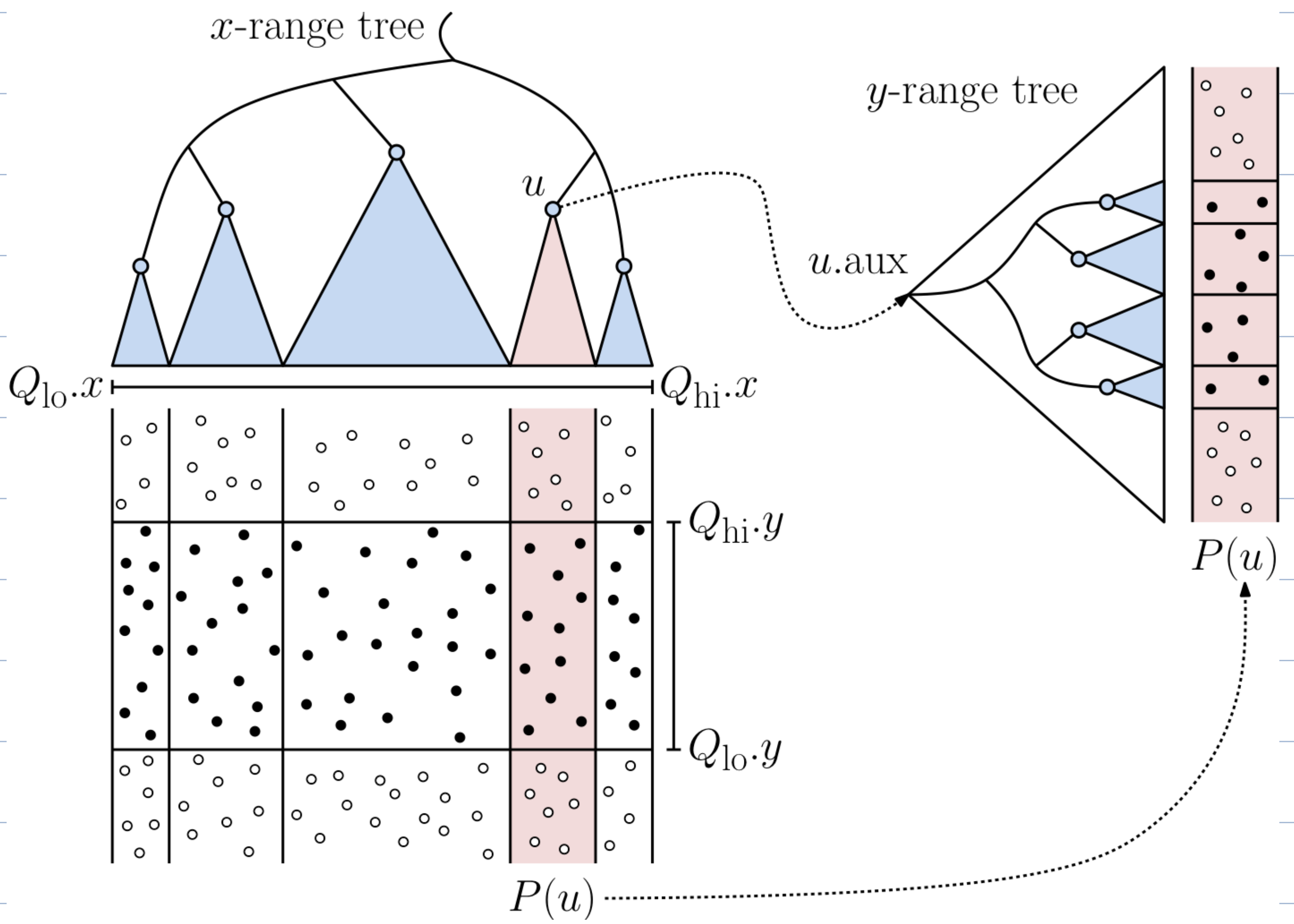
Orthogonal (2-d) Range Tree:

- Given points $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$
- Build a 1-d range tree for P based on x only (data in leaves)

Main tree

- For each internal node u, let $P(u)$ be points in its leaves (canon. subset)
  - Build a 1-d range tree for $P(u)$ sorted by y-coords.

Aux. tree for u



To process query $Q = [Q_{lo}, Q_{hi}]$
$= [Q_{lo.x}, Q_{hi.x}] \times [Q_{lo.y}, Q_{hi.y}]$

- Apply 1-d search in main tree with query $[Q_{lo.x}, Q_{hi.x}]$ to identify $O(\log n)$ maximal subtrees
- For each root u of one of these max. subtrees apply 1-d search in u.aux with query $[Q_{lo.y}, Q_{hi.y}]$
- Return overall sum

```
range2D(Node u, Range Q, Interval C = [x0, x1])
  if (u is leaf)
    return { 1  if u.point ∈ Q
           { 0  o.w.
  else if (Q.x ∩ C = ∅)        (no x overlap)
    return 0
  else if (C ⊆ Q.x)            (containment in x)
    return range1Dy(u.aux, Q, [-∞, +∞])    search aux. tree
  else                         (recurse)
    return range2D(u.left, Q, [x0, u.x])
         + range2D(u.right, Q, [u.x, x1])
```

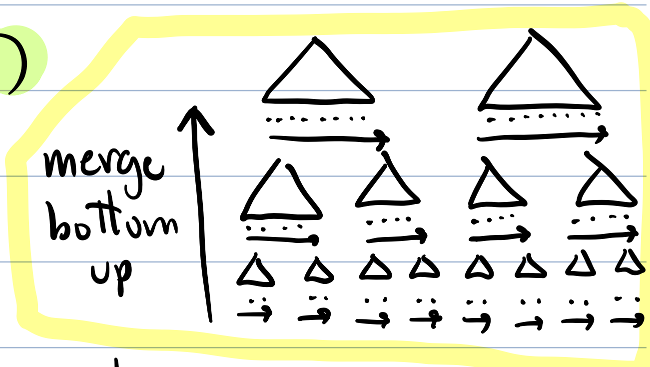
Space + Preprocessing Time:

- Since each node stores $O(1)$ data, total
  space = size of main tree + total size of aux. trees
- A tree with m leaves has size $O(m)$

$$\text{Space} = \underbrace{n}_{\text{main tree}} + \underbrace{\sum_u |P(u)|}_{u.\text{aux tree}}$$

- Main tree's height is $O(\log n)$
- Each leaf contributes a point to u.aux for each of its ancestors
  - $\Rightarrow$ Each point appears in $O(\log n)$ aux. trees
  - $\Rightarrow \sum_u |P(u)| = O(n \log n)$

$\Rightarrow$ Total space is $O(n \log n)$



Construction time:

Naive: $O(n \log^2 n)$

Better: Build aux trees bottom-up
- Two child sets can be merged in linear time

$\Rightarrow O(n \log n)$

Query Time:

Main tree: $O(\log n)$ time

→ Identifies $O(\log n)$ maximal subtrees
- each has $\leq n$ points
- each searchable in $O(\log n)$ time

⇒ Total time $= O(\log n) \cdot O(\log n)$
$= O(\log^2 n)$

Thm: Using orthogonal range trees, 2-dim orthog. range (counting) queries can be answered in:

$O(n \log n)$ space
$O(n \log n)$ build time
$O(\log^2 n)$ query time → $+k$ for reporting

Thm: Using orthogonal range trees, d-dim orthog. range (counting) queries can be answered in:

$O(n \log^{d-1} n)$ space
$O(n \log^{d-1} n)$ build time
$O(\log^d n)$ query time → $+k$ for reporting

Can we do better?

You can shave off a log n factor for query times - Cascading Search

2-dim: $O(\log^2 n) \to O(\log n)$
d-dim: $O(\log^d n) \to O(\log^{d-1} n)$

(See latex notes)

Idea:

- Final aux trees can be stored as sorted arrays (trees not needed)
- Always searching for same values: Q.lo.y Q.hi.y
- Can exploit knowledge of answer in one array to find answer in another, without doing search from scratch.