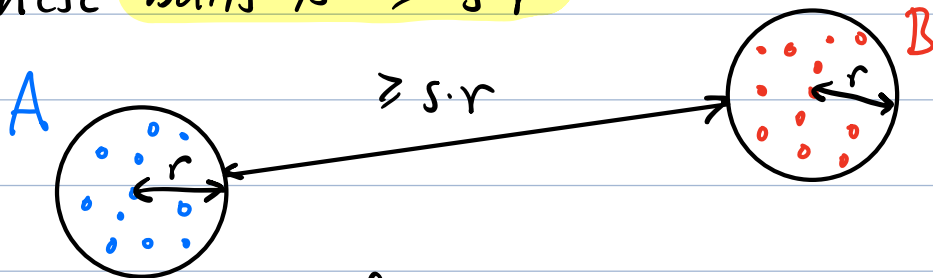


CMSC 754 - Computational Geometry

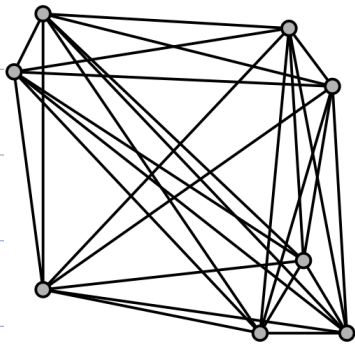
Lecture 17: Applications of WSPDs

Review of WSPDs:

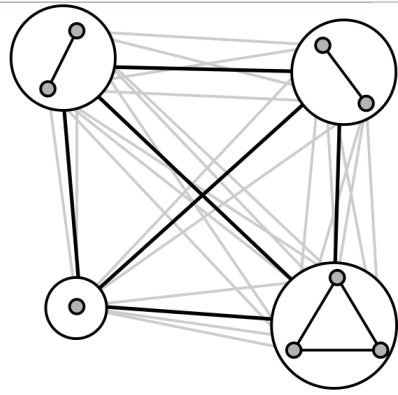
- Given a point set $P = \{p_1, \dots, p_n\}$ in \mathbb{R}^d (d a fixed constant) and separation factor $s > 0$, two sets $A + B$ are s -well separated if they can be contained in two balls of some radius r s.t. the distance between these balls is $\geq s \cdot r$



- An s -WSPD for P is a collection: $\{\{A_1, B_1\}, \dots, \{A_m, B_m\}\}$ such that
 - $A_i, B_i \subseteq P$
 - $A_i \cap B_i = \emptyset$ (disjoint)
 - $\cup_i A_i \otimes B_i = P \otimes P$ (cover all pairs)
 - $A_i + B_i$ are s -well separated
- Given $P + s \geq 1$, in time $O(n \log n + s \cdot n)$ we can construct an s -WSPD for P of size $O(s \cdot n)$.



28 pairs



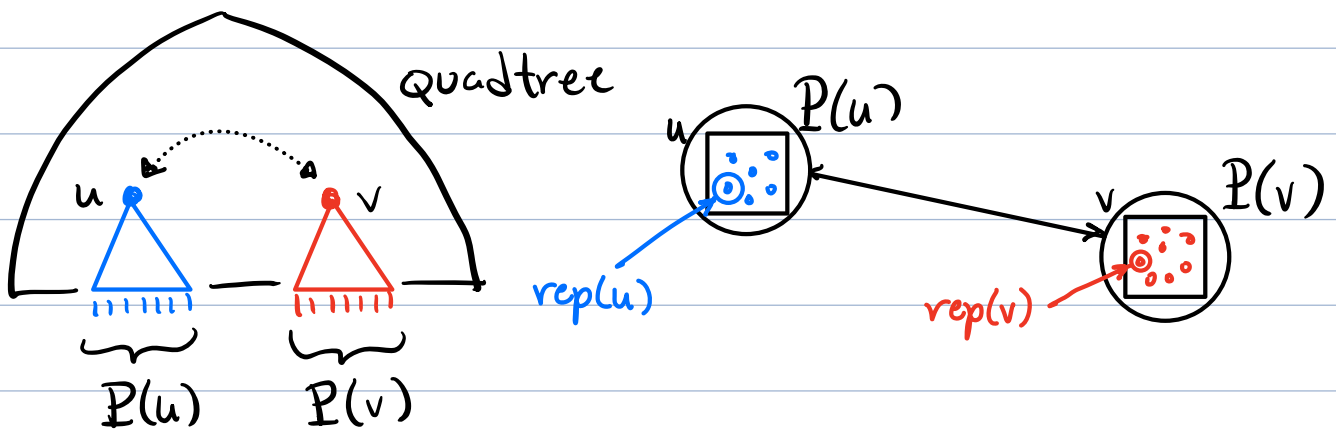
11 well-separated pairs

- Construction is based on **d-dim quad tree**

- Given nodes u, v in tree let

$P(u)$ - points in **u 's subtree**

$rep(u)$ - an **arbitrary pt of $P(u)$**
(u 's **representative**)



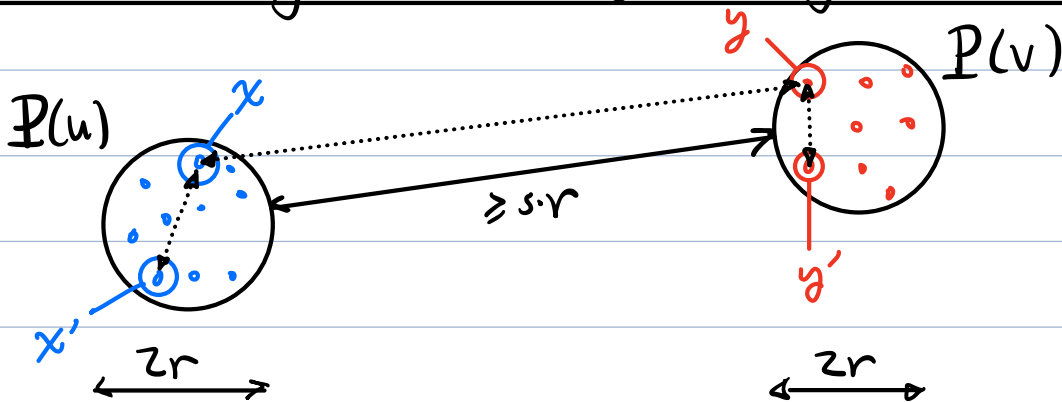
The WSP $\{P(u), P(v)\}$
is represented by the
pair $\{u, v\}$

Utility Lemma: Given an s -WSP $\{P(u), P(v)\}$

and $x, x' \in P(u) + y, y' \in P(v)$:

(i) $\|x - x'\| \leq \frac{2}{s} \cdot \|x - y\|$

(ii) $\|x' - y'\| \leq (1 + \frac{4}{s}) \cdot \|x - y\|$

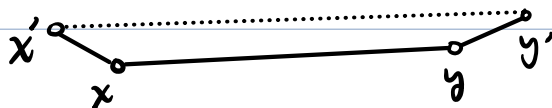


Intuition: (i) Same side closer than cross side
 (ii) Cross side dists similar

Proof: (i) $\|x - x'\| \leq 2 \cdot r$
 $= 2 \cdot r \frac{s \cdot r}{s \cdot r} \leq \frac{2 \cdot r}{s \cdot r} \|x - y\|$
 $= (\frac{2}{s}) \|x - y\| \quad \checkmark$

(ii) Observe: $\|x - y\| \geq s \cdot r \Rightarrow 4 \cdot r \leq \frac{4}{s} \|x - y\|$

By the triangle inequality:



$\|x' - y'\| \leq \|x' - x\| + \|x - y\| + \|y - y'\|$
 $\leq 2r + \|x - y\| + 2r$
 $\leq \|x - y\| + 4r$
 $\leq \|x - y\| + \frac{4}{s} \|x - y\|$
 $= (1 + \frac{4}{s}) \|x - y\| \quad \checkmark$

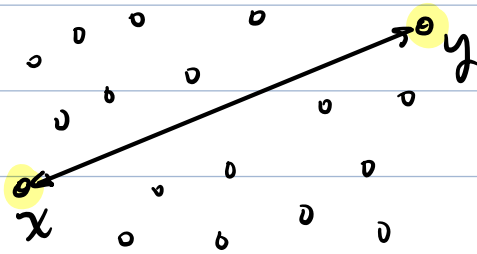
Applications:

- $(1+\epsilon)$ approx to diameter (farthest pair)
- exact closest pair
- Computing a t -spanner (for any $t > 1$)
- $(1+\epsilon)$ approx to Euclidean MST

$(1+\epsilon)$ Approx Diameter: in time $O(n \log n + \frac{n}{\epsilon^d})$

Given $P = \{p_1, \dots, p_n\}$ in \mathbb{R}^d

$$\text{diam}(P) = \max_{x, y \in P} \|x - y\|$$



Exact:

In \mathbb{R}^2 : Can compute in $O(n \log n)$

[Convex hull + rotating calipers]

\mathbb{R}^d : (Nearly) quadratic in n

$(1+\epsilon)$ -Approx:

- Set $s = 4/\epsilon$

- Compute an s -WSPD for P

- for each WSP $\{u, v\}$:

$$\text{dist}_{u,v} = \| \text{rep}(u) - \text{rep}(v) \|$$

- return $\max \text{dist}_{u,v}$ as approx diam

$O(n \log n + \frac{n}{\epsilon^d})$

$O(n/\epsilon^d)$

Correctness:

Plan:

① Since $\text{reps} \subseteq P$, $\text{approx diam} \leq \text{diam}(P)$

② We will show

*: $\exists \text{WSP } u, v \text{ s.t.}$

$$\text{dist}_{u,v} \geq \text{diam}(P)/(1+\epsilon)$$

$$\Rightarrow \max \text{dist}_{u,v} \geq \text{diam}(P)/(1+\epsilon)$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \frac{\text{diam}(P)}{1+\epsilon} \leq \text{approx diam} \leq \text{diam}(P) \quad \checkmark$$

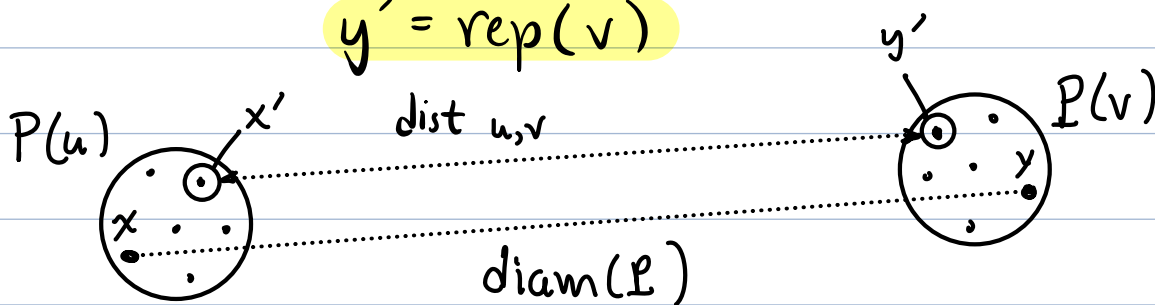
Need to show *

- Let x, y be diameter pair

- $\exists \text{WSP } \{u, v\}$ s.t. $x \in P(u)$ $y \in P(v)$

- Let $x' = \text{rep}(u)$

$y' = \text{rep}(v)$



By **WSPD utility lemma**:

$$\text{diam}(P) = \|x - y\| \leq \left(1 + \frac{4}{s}\right) \|x' - y'\|$$

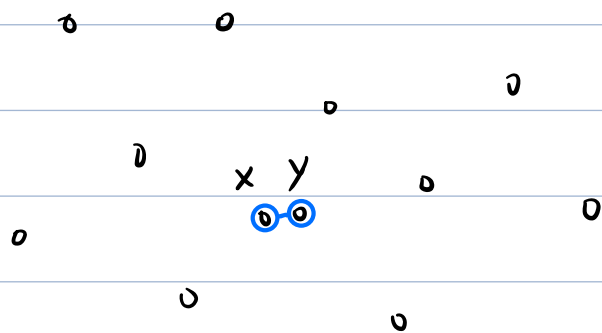
$$= (1 + \epsilon) \|x' - y'\| \quad (s = 4/\epsilon)$$

$$= (1 + \epsilon) \text{dist}_{u,v} \Rightarrow \text{dist}_{u,v} \geq \text{diam}(P)/(1 + \epsilon)$$

(Exact) Closest Pair: in time $O(n \log n)$

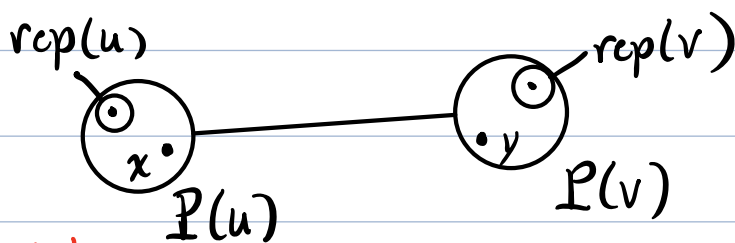
Given $P = \{p_1, \dots, p_n\}$ in \mathbb{R}^d find $x, y \in P$

$$\min_{x, y \in P} \|x - y\|$$



Intuition: Some WSP $\{u, v\}$ must cover the pair $\{x, y\}$

Huh? It looks like $x + y$ not closest!



It must be that $\text{rep}(u) = x + \text{rep}(v) = y$

Exact Closest Pair:

- Let $s > 2$ (eg. $s = 2.0001$)
- Build s -WSPD for P
- for each WSP $\{u, v\}$

$$\text{dist}_{u,v} = \|\text{rep}(u) - \text{rep}(v)\|$$

- return $\min_{u,v} \text{dist}_{u,v}$ as closest dist

$$\left. \begin{array}{l} O(n \log n + \\ 2^d \cdot n) \\ \end{array} \right\} = O(n \log n)$$

Correctness:

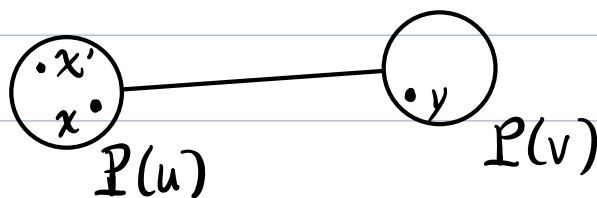
Follows directly from the following lemma:

Lemma: If $s > 0$ + x, y are closest pair in P , then any s -WSPD of P contains the pair $\{\{x\}, \{y\}\}$

That is, x, y are singletons in WSPD

Proof:

- Suppose not.
- Let $\{u, v\}$ be WSP with $x \in P(u), y \in P(v)$
- May assume w.l.o.g. that $P(u)$ has another pt x'



- By WSPD Utility Lemma:

$$\begin{aligned} \|x - x'\| &\leq \frac{2}{s} \cdot \|x - y\| \\ &< \|x - y\| \quad (\text{since } s > 2) \end{aligned}$$

$\Rightarrow x, y$ not closest pair
 \rightarrow contradiction

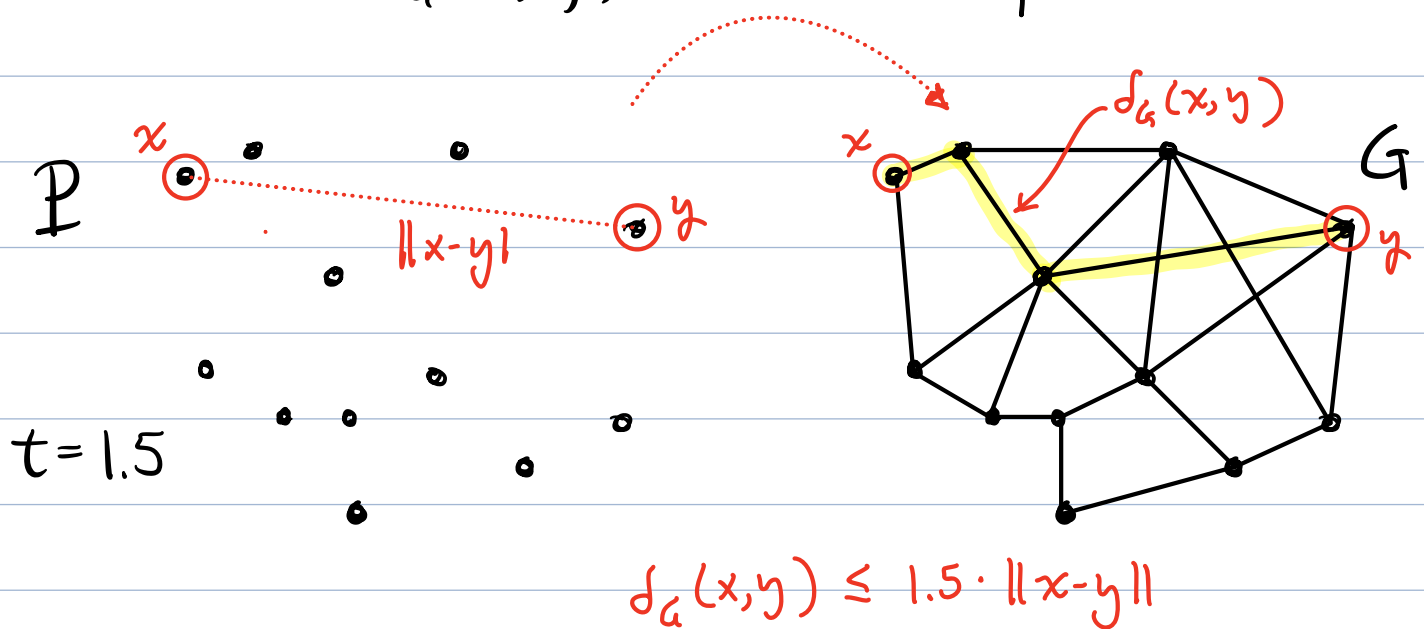
Spanners:

Recall def. of t -spanner (from lect. on Delaunay Tri.)

Given point set $P = \{p_1, \dots, p_n\}$ in \mathbb{R}^d and $t \geq 1$, a t -spanner is a graph on P s.t. $\forall x, y \in P$:

$$\|x-y\| \leq d_G(x,y) \leq t \cdot \|x-y\|$$

where $d_G(x,y)$ is shortest path dist in G



We will show that given $P \subseteq \mathbb{R}^d$ + $t > 1$ can build a $(1+\epsilon)$ -spanner for P in time $O(n \log n + n/\epsilon^d)$ consisting of $O(n/\epsilon^d)$ edges

Spanner construction (Given P + $t > 1$)

- Let $s = \frac{4(t+1)}{t-1}$
- $G \leftarrow$ graph with vertex set P + no edges
- Build an s -WSPD for P
- for each WSP $\{u, v\}$:
 - add edge $(\text{rep}(u), \text{rep}(v))$ to G
- return G

Time: If $t = 1 + \epsilon$, $s = O(1/\epsilon)$ [$0 < \epsilon < 1$]
 $\Rightarrow O(n \log n + n/\epsilon^2)$

Size: $O(n/\epsilon^2)$ WSPs $\Rightarrow O(n/\epsilon^2)$ edges

Correctness:

Will show that for all $x, y \in P$

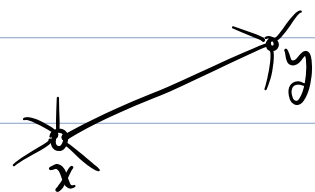
$$\|x - y\| \stackrel{\textcircled{1}}{\leq} d_G(x, y) \stackrel{\textcircled{2}}{\leq} t \cdot \|x - y\|$$

① Trivially true since G is a subgraph of complete Euclidean graph

② Rest of the proof...

Induction on num. of edges in path from x to y in G

Basis: Edge (x, y) is in G



$$\Rightarrow \delta_G(x, y) = \|x - y\| \leq t \cdot \|x - y\| \quad \checkmark$$

(since $t > 1$)

Induction step:

- \exists pair $\{u, v\}$ in WSPD that covers the pair (x, y)

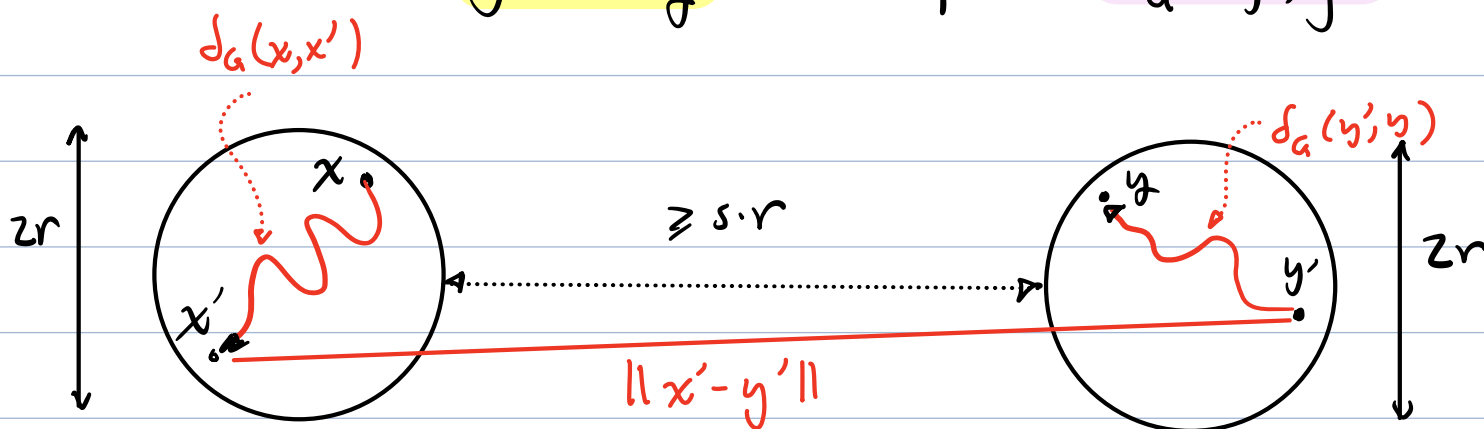
- Let $x' = \text{rep}(u)$ $y' = \text{rep}(v)$
(possibly $x' = x$ or $y' = y$)

- To get from x to y in G we can:

- x to x' \rightarrow path $\delta_G(x, x')$

- x' to y' \rightarrow direct edge: $\|x' - y'\|$

- y' to y \rightarrow path $\delta_G(y', y)$



By the induction hyp: $\delta_G(x, x') \leq t \cdot \|x - x'\|$

$\delta_G(y', y) \leq t \cdot \|y' - y\|$

$$\Rightarrow d_a(x, y) \leq t \cdot \|x - x'\| + \|x' - y'\| + t \cdot \|y' - y\|$$

$$= t(\|x - x'\| + \|y' - y\|) + \|x' - y'\|$$

By WSPD Utility Lemma:

- $\|x - x'\| \leq \frac{2}{s} \|x - y\|$
- $\|y' - y\| \leq \frac{2}{s} \|x - y\|$
- $\|x' - y'\| \leq \left(1 + \frac{4}{s}\right) \|x - y\|$

$$\Rightarrow d_a(x, y) \leq t \left(\frac{2}{s} \|x - y\| + \frac{2}{s} \|x - y\| \right) + \left(1 + \frac{4}{s}\right) \|x - y\|$$

$$= \left(t \frac{4}{s} + 1 + \frac{4}{s} \right) \|x - y\|$$

$$= \left(1 + \frac{4(t+1)}{s} \right) \|x - y\|$$

$$= t \|x - y\| \quad \left(\text{since: } s = \frac{4(t+1)}{t-1} \right)$$

□

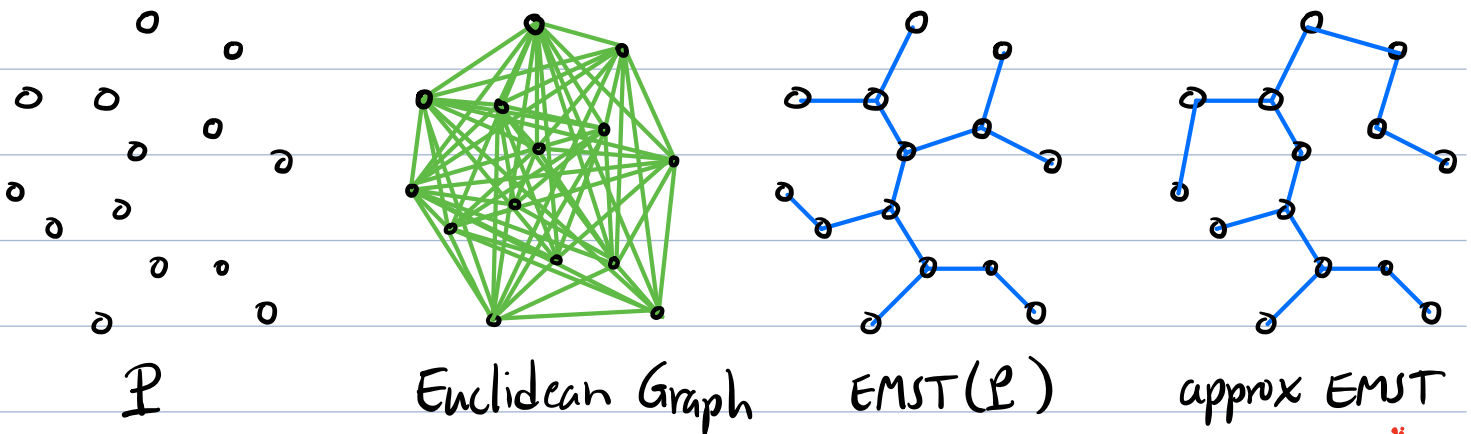
To obtain a $(1 + \epsilon)$ -spanner, set $t = 1 + \epsilon$ + apply this construction

Approx. to Euclidean MST

Given a point set $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$
define:

$EMST(P)$ = Min. spanning tree of
complete Euclidean graph
on P (where $w(u,v) = \|u-v\|$)

Let: $emst(P) = \sum_{(x,y) \in EMST(P)} \|x-y\|$
= total weight of $EMST(P)$



A graph H is an $(1+\epsilon)$ -approx $EMST$ if:

(1) H is a spanning tree for P

(2) $w(H) \leq (1+\epsilon) \cdot emst(P)$

where $w(H)$ = total weight of
 H 's edges

We'll show how to compute an $(1+\epsilon)$ -approx $EMST$
in time $O(n \log n + n/\epsilon^d)$

approx-EMST(P, ϵ)

- $G \leftarrow (1+\epsilon)$ -spanner for P
- return MST(G)

Time: Compute G : $O(n \log n + n/\epsilon^d)$

Compute MST(G):

- Can compute MST of a graph with v vertices + e edges in time

$$O(v \log v + e)$$

- G has n vertices + n/ϵ^d edges

- MST(G) takes $O(n \log n + n/\epsilon^d)$

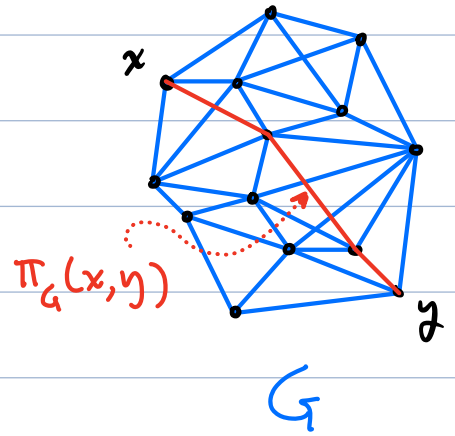
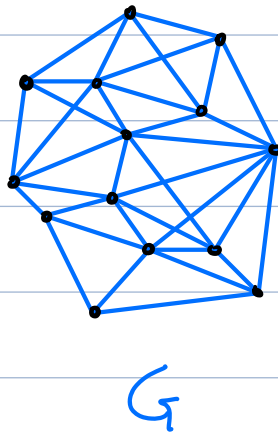
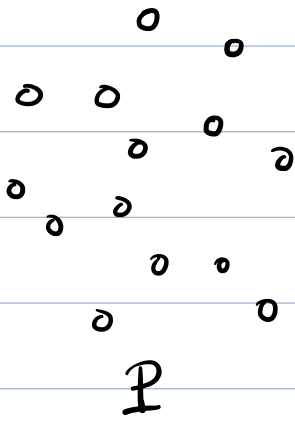
Correctness:

- We'll show that G has a connected subgraph that contains all pts of P ("spans P ") and has weight $\leq (1+\epsilon) \cdot \text{emst}(P)$
- If G has a spanning subgraph H of weight W , then the weight of its MST is no larger

For each $x, y \in P$, let $\pi_G(x, y)$ be shortest path from x to y in G .

Let $d_G(x, y)$ be length of this path

Know that $\delta_G(x, y) \leq (1 + \epsilon) \|x - y\|$



H :

for each $(x, y) \in \text{EMST}(P)$
add the edges of $\pi_G(x, y)$ to H

Obs:

① H is connected and spans all pts of P

② Total weight:

$$\omega(H) \leq \sum_{(x, y) \in \text{EMST}(P)} \delta_G(x, y)$$

$$\leq \sum_{(x, y) \in \text{EMST}(P)} (1 + \epsilon) \cdot \|x - y\|$$

$$= (1 + \epsilon) \sum_{(x, y) \in \text{EMST}(P)} \|x - y\|$$

$$= (1 + \epsilon) \cdot \text{emst}(P)$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \omega(\text{MST}(G))$$

$$\leq \omega(H) \leq (1 + \varepsilon) \cdot \text{emst}(P)$$

